Fluids, viscosity and strain energy

Upthrust, viscous drag and Stokes' law, terminal velocity in a fluid, and the elastic strain energy stored in a stretched material.

Upthrust and viscous drag

Upthrust

A body in a fluid feels an upward force, the **upthrust**, equal to the weight of fluid it displaces:

$$U = \rho_{\text{fluid}} V g$$

where V is the volume of fluid displaced. A body floats when its weight equals the upthrust.

> **TRY IT**
>
> A steel ball of volume 4.0×10^{-6} m³ is fully under oil of density 900 kg m^{-3}. Find the upthrust.

$U = 900 \times 4.0 \times 10^{-6} \times 9.81 = 3.5 \times 10^{-2}$ N.

Viscous drag

An object moving through a fluid feels a **viscous drag** that opposes the motion and grows with speed. For a small sphere moving slowly in a fluid, **Stokes' law** gives

$$F = 6\pi\eta r v$$

where η is the **viscosity** of the fluid, r the radius of the sphere and v its speed. The law holds for slow, smooth (laminar) flow only.

> **NOTE**
>
> Viscosity falls as temperature rises for a liquid such as oil. Hot oil is runnier.

Terminal velocity of a falling sphere

A sphere dropped into a viscous liquid speeds up until the forces balance. At **terminal velocity** the acceleration is zero:

$$\text{weight} = \text{upthrust} + \text{viscous drag}$$

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$$mg = U + 6\pi\eta r v_t$$

Writing the mass in terms of the sphere's density ρ_s and the liquid's density ρ_l gives

$$\tfrac{4}{3}\pi r^3(\rho_s - \rho_l)\,g = 6\pi\eta r v_t \quad \Rightarrow \quad v_t = \tfrac{2r^2(\rho_s-\rho_l)g}{9\eta}$$

At terminal velocity the resultant force is zero. The speed stops increasing, but the sphere keeps moving at constant speed.

TRY IT

A steel ball of radius 1.5 mm ($\rho_s = 7800$ kg m^{-3}) falls through a liquid of density 1200 kg m^{-3} at a terminal velocity of 0.060 m s^{-1}. Find the viscosity.

$$\eta = \tfrac{2r^2(\rho_s-\rho_l)g}{9v_t} = \tfrac{2\times(1.5\times10^{-3})^2\times6600\times9.81}{9\times0.060} = 0.54 \text{ Pa s}$$

The falling-ball method

Release a ball just below the surface and time it over a measured distance far enough down that it has reached terminal velocity. Repeat for balls of different radius. A graph of v_t against r^2 is a straight line through the origin, and the gradient gives η.

Stress, strain and energy stored

A material is described by:

- **Stress** $= \dfrac{F}{A}$ (Pa), the force per unit cross-sectional area.
- **Strain** $= \dfrac{\Delta x}{x}$ (no unit), the extension per unit original length.
- **Young modulus** $E = \dfrac{\text{stress}}{\text{strain}}$ (Pa), for a material that obeys Hooke's law.

Hooke's law gives $F = k\Delta x$ up to the **limit of proportionality**. Beyond the **elastic limit**, the material does not return to its original length.

Elastic strain energy

The work done in stretching a material is stored as **elastic strain energy**. It is the area under the force-extension graph:

$$E_{el} = \tfrac{1}{2}F\Delta x = \tfrac{1}{2}k(\Delta x)^2$$

When the graph is a curve, estimate the area by counting squares or splitting into strips.

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A wire obeying Hooke's law stretches by 2.0 mm under a force of 80 N. Find the energy stored.

$E_{el} = \frac{1}{2} \times 80 \times 2.0 \times 10^{-3} = 0.080$ J.

NOTE

For a loading and unloading pair of curves (a rubber band), the area between them is the energy dissipated as thermal energy.

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