Vectors

Vector notation and operations, magnitude and unit vectors, the equation of a line, whether lines meet, and the scalar product.

Vectors

Notation

A vector has size and direction. The same vector can be written in several ways:

$$\overrightarrow{AB} = \mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have length 1 and point along the x, y and z axes.

Operations

Add, subtract and scale component by component.

- $\mathbf{a} + \mathbf{b}$ is one vector followed by the other.
- $k\mathbf{a}$ is parallel to $\mathbf{a}$, and k times as long. A negative k reverses the direction.
- Two vectors are parallel if one is a multiple of the other.

The **position vector** of A is $\overrightarrow{OA}$, from the origin. Then:

> $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$. **The midpoint of AB has position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.**

> TRY IT
>
> $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$. C lies on AB with $AC : CB = 1 : 2$. Find $\overrightarrow{OC}$.

$\overrightarrow{AB} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, and $\overrightarrow{OC} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$:

$\overrightarrow{OC} = \frac{5}{3}\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k}$

Note Academy

Magnitude and unit vectors

$|x\mathbf{i}+y\mathbf{j}+z\mathbf{k}|=\sqrt{x^2+y^2+z^2}$. A unit vector in the direction of $\mathbf{a}$ is $\dfrac{\mathbf{a}}{|\mathbf{a}|}$.

TRY IT

Find the unit vector in the direction of $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

The magnitude is $\sqrt{4+1+4}=3$, so the unit vector is $\frac{1}{3}(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$.

The scalar product

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta=a_1b_1+a_2b_2+a_3b_3$$

Rearranged, this gives the angle between two vectors. If $\mathbf{a}\cdot\mathbf{b}=0$ for non-zero vectors, they are **perpendicular**.

TRY IT

Find the angle between $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}+\mathbf{j}-\mathbf{k}$.

$\cos\theta=\frac{1-1-2}{\sqrt{6}\sqrt{3}}=\frac{-2}{\sqrt{18}}\quad\Rightarrow\quad\theta=118.1°$

A negative scalar product means an obtuse angle. The acute angle between the lines is $180°-118.1°=61.9°$.

NOTE

For the angle at B in a triangle, both vectors must point away from B: use $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

The equation of a line

$\mathbf{r}=\mathbf{a}+t\mathbf{b}$, where $\mathbf{a}$ is the position vector of a point on the line, $\mathbf{b}$ is a direction vector, and t is a parameter that takes any real value.

Each value of t gives a different point on the line.

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Find the equation of the line through $A(1, 2, 3)$ and $B(2, 1, 5)$.

The direction is $\overrightarrow{AB} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, so

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

How two lines can meet

In three dimensions, two lines are in one of three positions:

Position	Test
Parallel	Direction vectors are multiples of each other
Intersecting	Not parallel, and a pair of parameter values satisfies all three components
Skew	Not parallel and they do not meet

TRY IT

Show that these lines meet, and find the point. l_1: $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and l_2: $\mathbf{r} = \mathbf{i} + 6\mathbf{k} + t(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

Equate components:

$1 + s = 1 + t \qquad 2 - s = t \qquad 3 + 2s = 6 - t$

The first two give $s = t = 1$. Check the third: $5 = 5$. The lines meet at $(2, 1, 5)$.

Solve two of the equations, then test the third. If the third fails and the lines are not parallel, they are skew. Showing the check is worth a mark.

The foot of a perpendicular

To find the point on a line closest to a point P, write a general point F on the line and make $\overrightarrow{PF}$ perpendicular to the direction.

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Find the foot of the perpendicular from $P(4, 3, 5)$ to l_1, and the shortest distance.

A general point on l_1 is $F(1 + s, 2 - s, 3 + 2s)$, so $\overrightarrow{PF} = (s - 3)\mathbf{i} + (-1 - s)\mathbf{j} + (2s - 2)\mathbf{k}$.

Using the scalar product with the direction:

$$(s - 3) - (-1 - s) + 2(2s - 2) = 0 \quad \Rightarrow \quad 6s - 6 = 0 \quad \Rightarrow \quad s = 1$$

The foot is $F(2, 1, 5)$. Then $\overrightarrow{PF} = -2\mathbf{i} - 2\mathbf{j}$, and the shortest distance is $\sqrt{8} = 2\sqrt{2}$.

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