Trigonometry

Graphs of sine, cosine and tangent, exact values, inverse notation and principal values, the two basic identities, and solving trigonometric equations in an interval.

Trigonometry

The three graphs

Graph	Period	Range	Notes
$y = \sin x$	360° or 2π	$-1 \leq y \leq 1$	Passes through the origin
$y = \cos x$	360° or 2π	$-1 \leq y \leq 1$	Starts at $y = 1$
$y = \tan x$	180° or π	All real values	Asymptotes at 90°, 270° and so on

The transformations of Functions apply here:

- $y = a \sin x$ has amplitude a.
- $y = \sin bx$ has period $\frac{360^\circ}{b}$, so there are b complete waves in 360°.
- $y = \sin x + c$ moves the wave up by c.
- $y = \sin(x + d)$ moves the wave left by d.

TRY IT

State the range of $y = 3 - 2\cos x$.

$\cos x$ runs from -1 to 1, so the expression runs from $3 - 2 = 1$ to $3 + 2 = 5$. The range is $1 \leq y \leq 5$.

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Exact values

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Related angles have the same size of value with a sign that depends on the quadrant. This diagram shows which ratios are positive in each quadrant:

So $\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$, and $\cos 120° = -\cos 60° = -\frac{1}{2}$.

Inverse notation and principal values

$\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ give one angle: the **principal value**, which is what the calculator shows.

Function	Principal values
$\sin^{-1} x$	$-90° \leq \theta \leq 90°$
$\cos^{-1} x$	$0° \leq \theta \leq 180°$
$\tan^{-1} x$	$-90° < \theta < 90°$

The two identities

$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ **and** $\sin^2\theta + \cos^2\theta \equiv 1$

The second can be rearranged as $\sin^2\theta \equiv 1 - \cos^2\theta$ and $\cos^2\theta \equiv 1 - \sin^2\theta$.

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Given $\sin\theta = \frac{3}{5}$ and that θ is obtuse, find $\cos\theta$ and $\tan\theta$.

$\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$. Cosine is negative for an obtuse angle, so $\cos\theta = -\frac{4}{5}$ and $\tan\theta = -\frac{3}{4}$.

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Prove that $\tan\theta + \dfrac{1}{\tan\theta} \equiv \dfrac{1}{\sin\theta\cos\theta}$.

$\text{LHS} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} = \text{RHS}$

NOTE

In a proof, start from one side and work to the other. Do not treat the identity as an equation and do the same thing to both sides.

Solving trigonometric equations

There are usually several solutions in an interval. Find the principal value, then use the symmetry of the graph:

Equation	Second solution from α	Then
$\sin x = k$	$180° - \alpha$	Add or subtract 360°
$\cos x = k$	$360° - \alpha$	Add or subtract 360°
$\tan x = k$	$180° + \alpha$	Add or subtract 180°

TRY IT

Solve $\sin x = 0.5$ for $0° \le x \le 360°$.

$x = 30°$ and $x = 180° - 30° = 150°$.

When the angle is changed

Change the interval first, solve for the whole bracket, and only then find x.

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Solve $\cos 2x = 0.6$ for $0° \le x \le 180°$.

The interval becomes $0° \le 2x \le 360°$. The solutions are $2x = 53.13°$ and $2x = 306.87°$, so $x = 26.6°$ or $x = 153.4°$.

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Solve $\tan(x - 30°) = 1$ for $0° \le x \le 360°$.

The interval becomes $-30° \le x - 30° \le 330°$. So $x - 30° = 45°$ or $225°$, giving $x = 75°$ or $x = 255°$.

Using the identities in equations

Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ when an equation has both sine and cosine to the same power.

TRY IT

Solve $3\sin x = 4\cos x$ for $0° \le x \le 360°$.

Divide by $\cos x$: $\tan x = \dfrac{4}{3}$, so $x = 53.1°$ or $x = 233.1°$.

Use $\sin^2\theta + \cos^2\theta = 1$ when squares appear, to get a quadratic in one ratio.

TRY IT

Solve $2\sin^2 x + 3\cos x = 0$ for $0° \le x \le 360°$.

$2(1 - \cos^2 x) + 3\cos x = 0 \quad \Rightarrow \quad 2\cos^2 x - 3\cos x - 2 = 0 \quad \Rightarrow \quad (2\cos x + 1)(\cos x - 2) = 0$

$\cos x = 2$ is impossible, so $\cos x = -\dfrac{1}{2}$ and $x = 120°$ or $x = 240°$.

Never cancel a trigonometric factor. In $\sin x\cos x = \sin x$, dividing by $\sin x$ loses the solutions of $\sin x = 0$. Factorise instead: $\sin x(\cos x - 1) = 0$.

NOTE

Give the answer in the form the question uses. If the interval is in radians, the answers must be in radians too.

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