Series

The binomial expansion for a positive integer power, arithmetic and geometric progressions, their sums, and the sum to infinity of a convergent geometric progression.

Series

The binomial expansion

For a positive integer n:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n$$

where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$. The term containing b^r is

$\binom{n}{r}a^{n-r}b^r$**. The two powers always add to** n**.**

TRY IT

Find the coefficient of x^2 in the expansion of $(2+x)^5$.

$$\binom{5}{2} \times 2^3 \times x^2 = 10 \times 8 \times x^2 = 80x^2$$

The coefficient is 80. Put any negative sign or coefficient of x inside the bracket: in $(2-3x)^5$, the term has $(-3x)^2 = 9x^2$.

Finding a particular term

When the terms include powers of x in both parts, write the general term and collect the powers.

TRY IT

Find the term independent of x in $\left(x + \dfrac{2}{x}\right)^6$.

The general term is $\binom{6}{r}x^{6-r}\left(\dfrac{2}{x}\right)^r = \binom{6}{r}2^r x^{6-2r}$.

"Independent of x" means the power is zero, so $r = 3$, and the term is $20 \times 8 = 160$.

Note Academy

Multiplying by another bracket

TRY IT

Find the coefficient of x^2 in $(1-2x)(2+x)^5$.

$(2+x)^5 = 32 + 80x + 80x^2 + \ldots$

The x^2 terms come from $1 \times 80x^2$ and $-2x \times 80x$. So the coefficient is $80 - 160 = -80$.

Arithmetic progressions

An **arithmetic progression** (AP) adds a common difference d each time. With first term a:

$u_n = a + (n-1)d$ **and** $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$**, where** l **is the last term.**

TRY IT

The 5th term of an AP is 17 and the 12th term is 45. Find the sum of the first 20 terms.

$a + 4d = 17$ and $a + 11d = 45$. Subtracting gives $7d = 28$, so $d = 4$ and $a = 1$.

$S_{20} = \frac{20}{2}[2 + 19 \times 4] = 10 \times 78 = 780$

Geometric progressions

A **geometric progression** (GP) multiplies by a common ratio r each time:

$u_n = ar^{n-1}$ **and** $S_n = \frac{a(1-r^n)}{1-r}$ **for** $r \neq 1$**.**

TRY IT

The 3rd term of a GP is 12 and the 6th term is 96. Find the sum of the first 10 terms.

$ar^2 = 12$ and $ar^5 = 96$. Dividing gives $r^3 = 8$, so $r = 2$ and $a = 3$.

$S_{10} = \frac{3(1-2^{10})}{1-2} = 3 \times 1023 = 3069$

Note Academy

Telling them apart

Three consecutive terms x, y, z are:

- in an AP if $y - x = z - y$
- in a GP if $\frac{y}{x} = \frac{z}{y}$

> **TRY IT**
>
> k, $k + 6$ and $3k$ are consecutive terms of a GP with $k > 0$. Find k.

$\frac{k+6}{k} = \frac{3k}{k+6} \quad \Rightarrow \quad (k+6)^2 = 3k^2 \quad \Rightarrow \quad k^2 - 6k - 18 = 0$

$k = \frac{6 + \sqrt{108}}{2} = 3 + 3\sqrt{3}$, taking the positive root.

Sum to infinity

When $|r| < 1$, the powers of r shrink towards zero, so the sum settles at a limit. The GP is **convergent**.

> $S_\infty = \frac{a}{1 - r}$, **valid only when** $|r| < 1$.

> **TRY IT**
>
> A GP has first term 5 and sum to infinity 20. Find the common ratio.

$\frac{5}{1 - r} = 20$, so $1 - r = \frac{1}{4}$ and $r = \frac{3}{4}$.

Common forms of question

- **Mixed AP and GP.** Set up one equation from each, then solve them together. Reject a value of r with $|r| \geq 1$ when a sum to infinity exists.
- **Least value of n.** Form an inequality from the sum, then try values of n or use logarithms. Dividing by $\ln r$ when $r < 1$ reverses the inequality, because that logarithm is negative.
- **Real contexts.** Savings or lengths growing by a fixed amount are an AP. Growth by a fixed percentage is a GP with $r = 1 + \frac{p}{100}$.

> **NOTE**

Note Academy

Check whether the question wants the nth term or the sum of n terms. It is the most common slip in this topic.

Note Academy