Quadratics

Completing the square, the discriminant, quadratic equations and inequalities, simultaneous equations with one linear and one quadratic, and equations that are quadratic in a function of x.

Quadratics

Completing the square

Completing the square rewrites $ax^2 + bx + c$ as $a(x + p)^2 + q$. This form shows the turning point of the curve straight away.

For $x^2 + bx + c$, halve the coefficient of x, then subtract the square of what you halved:

$$x^2 + bx + c = (x + \tfrac{b}{2})^2 - (\tfrac{b}{2})^2 + c$$

> **TRY IT**
>
> Write $x^2 + 6x + 11$ in the form $(x + p)^2 + q$.

$$x^2 + 6x + 11 = (x + 3)^2 - 9 + 11 = (x + 3)^2 + 2$$

When $a \neq 1$, take a out of the first two terms only.

> **TRY IT**
>
> Write $2x^2 - 8x + 3$ in the form $a(x + p)^2 + q$.

$$2x^2 - 8x + 3 = 2(x^2 - 4x) + 3 = 2\left[(x - 2)^2 - 4\right] + 3 = 2(x - 2)^2 - 5$$

What the completed square tells you

For $y = a(x + p)^2 + q$, the squared bracket is never negative, so:

Value of a	Shape	Turning point	Range of y
$a > 0$	U-shaped	Minimum at $(-p, q)$	$y \geq q$
$a < 0$	n-shaped	Maximum at $(-p, q)$	$y \leq q$

So $y = 2(x - 2)^2 - 5$ has a minimum at $(2, -5)$, its line of symmetry is $x = 2$, and its range is $y \geq -5$.

Note Academy

You will use this again in Functions: the completed square gives the range of a quadratic function and the value of k that makes it one-one on $x \geq k$.

The discriminant

For $ax^2 + bx + c = 0$, the quadratic formula is

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The part under the root, $b^2 - 4ac$, is the **discriminant**. It tells you how many real roots there are, and so how the curve meets the x-axis.

Discriminant	Roots	The curve
$b^2 - 4ac > 0$	Two distinct real roots	Crosses the axis twice
$b^2 - 4ac = 0$	One repeated root	Touches the axis
$b^2 - 4ac < 0$	No real roots	Misses the axis

TRY IT

Find the values of k for which $x^2 + kx + 9 = 0$ has two distinct real roots.

$k^2 - 4(1)(9) > 0 \quad \Rightarrow \quad k^2 > 36 \quad \Rightarrow \quad k < -6 \text{ or } k > 6$

NOTE

"Real roots" with no other word includes the repeated case, so the condition is $b^2 - 4ac \geq 0$. "Two distinct" needs > 0.

Quadratic inequalities

Solve the equation first, then use a sketch to pick the correct side.

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Solve $x^2 - x - 6 > 0$.

$(x - 3)(x + 2) > 0$. The U-shaped curve crosses the axis at $x = -2$ and $x = 3$, and is above the axis outside these values:

Note Academy

$x < -2 \quad \text{or} \quad x > 3$

> **TRY IT**
>
> Solve $2x^2 + x - 3 \leq 0$.

$(2x + 3)(x - 1) \leq 0$. The curve is on or below the axis between the roots:

$-\frac{3}{2} \leq x \leq 1$

> **For a U-shaped curve, "> 0" gives two separate regions and "< 0" gives one region between the roots. Write two separate regions with "or", never as $3 < x < -2$.**

Simultaneous equations

When one equation is linear and the other quadratic, make x or y the subject of the linear one and substitute.

> **TRY IT**
>
> Solve $x + y = 5$ and $x^2 + y^2 = 13$.

Put $y = 5 - x$ into the second equation:

$x^2 + (5 - x)^2 = 13 \quad \Rightarrow \quad 2x^2 - 10x + 12 = 0 \quad \Rightarrow \quad (x - 2)(x - 3) = 0$

So $x = 2, y = 3$ or $x = 3, y = 2$. Always give the pairs together.

Lines and curves

Setting a line equal to a curve gives a quadratic, and its discriminant says how they meet:

Discriminant	The line
> 0	Cuts the curve at two points
$= 0$	Is a tangent to the curve
< 0	Does not meet the curve

> **TRY IT**

Note Academy

Find the value of k for which $y = 2x + k$ is a tangent to $y = x^2 - 4x + 5$.

$x^2 - 4x + 5 = 2x + k \quad \Rightarrow \quad x^2 - 6x + 5 - k = 0$

For a tangent, $36 - 4(5 - k) = 0$, so $16 + 4k = 0$ and $k = -4$.

Equations that are quadratic in a function of x

Some equations are quadratics in disguise. Replace the function with a single letter, solve, then undo the substitution.

TRY IT

Solve $x^4 - 5x^2 + 4 = 0$.

Let $u = x^2$: $u^2 - 5u + 4 = 0$, so $u = 1$ or $u = 4$. Then $x^2 = 1$ or $x^2 = 4$, giving $x = \pm 1$ or $x = \pm 2$.

TRY IT

Solve $x - 5\sqrt{x} + 6 = 0$.

Let $u = \sqrt{x}$: $u^2 - 5u + 6 = 0$, so $u = 2$ or $u = 3$. Then $x = 4$ or $x = 9$.

NOTE

Always finish in terms of x, and check the substitution allows each value. $\sqrt{x} = -2$ would have no solution, because the root is never negative.

The same idea handles $2\sin^2 x + \sin x - 1 = 0$ with $u = \sin x$ and $3^{2x} - 4(3^x) + 3 = 0$ with $u = 3^x$.

Note Academy