Proof

What a proof must do, proof by exhaustion, disproof by counter-example, proof by contradiction, and proof by mathematical induction.

Proof by exhaustion and counter-example

A **proof** starts from stated assumptions and moves by logical steps to a conclusion. Every step must follow from the one before. Checking a few cases is not a proof.

Proof by exhaustion

If a statement concerns a small, finite set of cases, check every one.

> **TRY IT**
>
> Show that every square number is of the form $3k$ or $3k + 1$, where k is an integer.

Every integer is $3n$, $3n + 1$ or $3n + 2$, so check all three.

$(3n)^2 = 9n^2 = 3(3n^2)$

$(3n + 1)^2 = 9n^2 + 6n + 1 = 3(3n^2 + 2n) + 1$

$(3n + 2)^2 = 9n^2 + 12n + 4 = 3(3n^2 + 4n + 1) + 1$

Every case gives $3k$ or $3k + 1$, so the statement is true for all integers.

> **NOTE**
>
> The cases must cover every possibility. Say why they do: here, dividing by 3 leaves remainder 0, 1 or 2 and nothing else.

Direct proof with algebra

For a statement about all integers, let a general integer stand for any value. An even number is $2n$ and an odd number is $2n + 1$.

> **TRY IT**
>
> Prove that the sum of the squares of two consecutive integers is odd.

Note Academy

Let the integers be n and $n+1$.

$n^2 + (n+1)^2 = 2n^2 + 2n + 1 = 2(n^2 + n) + 1$

This is $2k+1$ with $k = n^2 + n$ an integer, so it is odd.

Disproof by counter-example

One case where a statement fails is enough to show it is false. This is the only way a single example can settle a question.

> **TRY IT**
>
> Disprove the statement "$n^2 + n + 11$ is prime for every positive integer n".

Try values until one fails. When $n = 11$:

$11^2 + 11 + 11 = 143 = 11 \times 13$

This is not prime, so the statement is false.

> **To disprove "for all n", give one value of n for which the statement is false, and show the working that shows it is false. Examples that work prove nothing.**

Proof by contradiction

To prove a statement, assume it is false and show that this leads to something impossible. The assumption must then be wrong.

> **TRY IT**
>
> Prove that $\sqrt{2}$ is irrational.

Assume $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{p}{q}$ where p and q are integers with no common factor.

Squaring, $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$. Hence p^2 is even, so p is even. Write $p = 2m$.

Then $4m^2 = 2q^2$, so $q^2 = 2m^2$. Hence q^2 is even, so q is even.

Now p and q are both even, which contradicts "no common factor". The assumption was false, so $\sqrt{2}$ is irrational.

> **TRY IT**

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Prove that there is no greatest odd integer.

Assume there is a greatest odd integer N. Then $N+2$ is also an odd integer, and it is larger than N. This contradicts N being greatest, so there is no greatest odd integer.

Writing a contradiction proof

1. State the opposite of what is to be proved: "Assume that ... ".
2. Work forward with correct steps until two statements disagree, or a statement is plainly false.
3. Conclude: "This is a contradiction, so the assumption is false and the original statement is true."

NOTE

Be careful with the opposite. The opposite of "x is rational" is "x is irrational". The opposite of "all n satisfy P" is "at least one n does not".

Proof by induction

NOTE

Proof by induction is in the Further Pure units, not P1 to P4. Learn it if you take Further Mathematics.

Induction proves a statement $P(n)$ for every positive integer n in three parts.

1. **Basis.** Show $P(1)$ is true.
2. **Assumption.** Assume $P(k)$ is true for some positive integer k.
3. **Inductive step.** Using the assumption, show $P(k+1)$ is true.

Then $P(1)$ is true, so $P(2)$ is true, so $P(3)$ is true, and so on. The conclusion is that $P(n)$ is true for all positive integers n.

A sum

TRY IT

Prove that $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ for all positive integers n.

Basis. When $n=1$, the left side is 1 and the right side is $\dfrac{1\times 2}{2}=1$. It is true for $n=1$.

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Assumption. Suppose it is true for $n = k$:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Step. For $n = k + 1$, add the next term to both sides:

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

This is the formula with $n = k + 1$.

Conclusion. It is true for $n = 1$. If it is true for $n = k$ then it is true for $n = k + 1$. Hence it is true for all positive integers n.

Divisibility

For a statement such as "$f(n)$ is divisible by d", write $f(k+1)$ in terms of $f(k)$ so that the assumption can be used.

> **TRY IT**
>
> Prove that $7^n - 1$ is divisible by 6 for all positive integers n.

Basis. For $n = 1$: $7 - 1 = 6$, which is divisible by 6.

Assumption. Suppose $7^k - 1$ is divisible by 6, so $7^k - 1 = 6m$ for an integer m.

Step.

$$7^{k+1} - 1 = 7 \times 7^k - 1 = 7(6m + 1) - 1 = 42m + 6 = 6(7m + 1)$$

This is divisible by 6.

Conclusion. True for $n = 1$, and true for $n = k + 1$ whenever it is true for $n = k$, so true for all positive integers n.

Writing the conclusion

Always finish with a sentence that says all three things: true for $n = 1$, true for $n = k + 1$ if true for $n = k$, so true for all positive integers n. Most of the marks are lost by leaving the basis or the conclusion out.

Note Academy