Parametric equations

Curves given by x and y in terms of a parameter, converting between parametric and Cartesian forms, and finding points on a parametric curve.

Parametric equations

A curve can be given by two equations that give x and y in terms of a third variable t, called the **parameter**:

$$x = f(t), \qquad y = g(t)$$

Each value of t gives one point (x, y). As t changes, the point moves along the curve.

Points on the curve

TRY IT

A curve has parametric equations $x = 2t$, $y = t^2 - 1$. Find the points where $t = 0$, $t = 1$ and $t = -2$.

t	$x = 2t$	$y = t^2 - 1$	**Point**
0	0	-1	$(0, -1)$
1	2	0	$(2, 0)$
-2	-4	3	$(-4, 3)$

To find where the curve crosses an axis, set $x = 0$ or $y = 0$, solve for t, and use that t in the other equation.

TRY IT

For $x = 2t$, $y = t^2 - 1$, find where the curve crosses the x-axis.

On the x-axis $y = 0$, so $t^2 = 1$ and $t = \pm 1$. Then $x = \pm 2$. The curve crosses the x-axis at $(2, 0)$ and $(-2, 0)$.

Converting to Cartesian form

To remove the parameter, make t the subject of one equation and substitute it into the other.

TRY IT

Note Academy

Find the Cartesian equation of the curve $x = 2t, y = t^2 - 1$.

From $x = 2t$, $t = \frac{x}{2}$. Substitute:

$y = (\frac{x}{2})^2 - 1 = \frac{x^2}{4} - 1$

This is a parabola.

When both equations involve trigonometric functions of the parameter, use an identity such as $\sin^2\theta + \cos^2\theta = 1$ instead.

TRY IT

A curve has equations $x = 3\cos\theta, y = 2\sin\theta$. Find its Cartesian equation.

$\cos\theta = \frac{x}{3}$ and $\sin\theta = \frac{y}{2}$. Then

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

This is an ellipse.

To eliminate the parameter: make it the subject and substitute, or use an identity ($\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$) to join the two equations.

Restrictions on the parameter

If the parameter has a restricted range, only part of the Cartesian curve is traced out. State the range of x or y that the parameter allows.

TRY IT

A curve has equations $x = t^2, y = 2t$ for $t \geqslant 0$. Find its Cartesian equation and state the range of x.

From $y = 2t$, $t = \frac{y}{2}$, so $x = \frac{y^2}{4}$, or $y^2 = 4x$. Since $t \geqslant 0, y \geqslant 0$ and $x \geqslant 0$. Only the upper half of the parabola is on the curve.

Converting from Cartesian form

A Cartesian curve can be given a parameter by choosing one of x or y to equal a simple expression in t.

Note Academy

TRY IT

Write $y = x^2 + 3$ in parametric form using $x = t$.

With $x = t, y = t^2 + 3$.

NOTE

A curve has many parametric forms. Use the one the question gives, or the simplest that makes the working short.

Parametric equations in problems

Problems often ask for where two curves meet, or for a point on a curve with a given property. Substitute the parametric equations into the other curve's equation and solve for t.

TRY IT

The curve $x = t^2, y = 2t$ meets the line $y = x - 3$. Find the values of t at the points of intersection.

Substitute: $2t = t^2 - 3$, so $t^2 - 2t - 3 = 0$, which gives $(t - 3)(t + 1) = 0$ and $t = 3$ or $t = -1$. The points are $(9, 6)$ and $(1, -2)$.

Note Academy