Numerical solution of equations

Locating a root with a graph or a sign change, and using an iterative formula to find a root to a required accuracy.

Numerical solution of equations

Why numerical methods

Many equations, such as $x^3 + x - 3 = 0$ or $e^x = 3 - x$, have no neat algebraic solution. A numerical method finds a root to whatever accuracy is needed.

Locating a root from a graph

The number of roots of $f(x) = g(x)$ is the number of times $y = f(x)$ and $y = g(x)$ cross. Sketching the two graphs shows how many roots there are and roughly where.

> **TRY IT**
>
> By sketching suitable graphs, show that $e^x = 3 - x$ has exactly one root.

$y = e^x$ is always increasing and $y = 3 - x$ is always decreasing, so they cross exactly once.

Locating a root by a sign change

If f is continuous and $f(a)$ and $f(b)$ have opposite signs, there is a root between a and b.

> **TRY IT**
>
> Show that $x^3 + x - 3 = 0$ has a root between 1 and 2.

$f(1) = -1 < 0$ and $f(2) = 7 > 0$. The sign changes and f is continuous, so there is a root in the interval.

> **A complete answer states both values, says that the sign changes, and concludes that there is a root. Rearrange the equation to $f(x) = 0$ first.**

> **NOTE**

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A sign change does not work where a function has an asymptote: $\tan x$ changes sign across 90° without a root. It also misses a pair of roots close together, or a repeated root where the curve only touches the axis.

Iterative formulae

An **iterative formula** $x_{n+1} = \mathrm{F}(x_n)$ uses each value to produce the next. If the sequence converges to a limit α, then $\alpha = \mathrm{F}(\alpha)$, so the limit is a root of $x = \mathrm{F}(x)$.

TRY IT

Show that the limit of $x_{n+1} = (3 - x_n)^{\frac{1}{3}}$ is a root of $x^3 + x - 3 = 0$.

At the limit, $x = (3 - x)^{\frac{1}{3}}$. Cubing gives $x^3 = 3 - x$, so $x^3 + x - 3 = 0$.

Using an iteration

Store each answer and use the calculator's answer key, so that no accuracy is lost. Keep at least two more decimal places than the question asks for.

TRY IT

Use $x_{n+1} = (3 - x_n)^{\frac{1}{3}}$ with $x_1 = 1$ to find the root to 2 decimal places. Give each iterate to 4 decimal places.

$x_2 = 1.2599 \quad x_3 = 1.2028 \quad x_4 = 1.2158 \quad x_5 = 1.2129 \quad x_6 = 1.2135 \quad x_7 = 1.2134$

The root is 1.21 to 2 decimal places.

Show every iterate, to the number of places the question asks for, and stop when two successive values round to the same answer.

Checking the accuracy

To prove that 1.21 is correct to 2 decimal places, test the two ends of the rounding interval:

$\mathrm{f}(1.205) = -0.045 < 0 \qquad \mathrm{f}(1.215) = 0.009 > 0$

The sign changes, so the root lies between 1.205 and 1.215, and it rounds to 1.21.

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Common forms of question

Questions in this topic often follow the same path.

1. Set up an equation from a context, often from a sector, a triangle or a stationary point: "Show that α satisfies $x = \ldots$".
2. Show by a sign change that the root lies in a given interval.
3. Use the iteration to find it, with each step shown.

> **NOTE**
>
> Not every rearrangement converges. A question will normally give you one that does. Near the root, it converges when the gradient of F is between -1 and 1.

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