Models, vectors and moments in mechanics

Mathematical modelling and the standard modelling words in mechanics, vectors for displacement, velocity, acceleration and force in a plane, and the moment of a force.

Mathematical models in mechanics

A **mathematical model** replaces a real situation with a simpler one that can be solved. Mechanics models make assumptions and state them in standard words. Know what each word lets you do.

Word	What it means	What it lets you do
Particle	All the mass is at one point; size is ignored	No turning effect, no air resistance on the shape
Light	Mass is negligible	Treat its weight as zero; tension is the same all along a light string
Inextensible string	Does not stretch	Connected objects have the same speed and acceleration
Smooth	No friction	Contact force is perpendicular to the surface
Rough	Friction acts	Friction opposes motion or tendency to move
Rigid	Does not bend	A rod or beam keeps its shape
Uniform	Mass is spread evenly	The centre of mass is at the middle
Thin / wire	Width ignored	A rod or ring is treated as a line
Small pulley	Smooth and light	Tension is the same on both sides
Constant gravity	g does not change	Weight is mg; use $g = 9.8$ or 9.81 as the question says
Air resistance ignored	Only gravity acts on a projectile	Constant acceleration g downwards

NOTE

When a question asks how a model could be improved or what an assumption means, link the word to its effect: "Treating the string as light means the tension is the same at both ends."

Note Academy

Modelling comments

TRY IT

A ball is modelled as a particle falling under gravity alone. State one way the model is unrealistic and what it would change.

Air resistance is ignored. In reality it opposes the motion, so the real acceleration would be smaller than g and the real speed at the bottom would be less than the model predicts.

State the assumption and its effect on the answer: what in the model is wrong, and whether the answer would be bigger or smaller.

Vectors in mechanics

Displacement, velocity, acceleration and force are vectors. In a plane, write them using unit vectors $\mathbf{i}$ and $\mathbf{j}$ in two perpendicular directions, or as column vectors.

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} \quad \text{or} \quad \mathbf{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Magnitude and direction

For $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$:

$$|\mathbf{v}| = \sqrt{a^2 + b^2}, \qquad \tan\theta = \frac{b}{a}$$

where θ is the angle with the $\mathbf{i}$ direction. Draw a sketch: the signs of a and b tell you the quadrant.

TRY IT

A particle has velocity $(3\mathbf{i} - 4\mathbf{j})$ m s^{-1}. Find its speed and the angle its direction makes with $\mathbf{i}$.

Speed $= \sqrt{3^2 + 4^2} = 5$ m s^{-1}. The angle below $\mathbf{i}$ is $\tan^{-1}\dfrac{4}{3} = 53.1°$.

Resultant forces and Newton's second law

The resultant of several forces is their vector sum. For constant mass:

$$\mathbf{F} = m\mathbf{a}$$

TRY IT

Note Academy

Forces $(2\mathbf{i}+5\mathbf{j})$ N and $(4\mathbf{i}-\mathbf{j})$ N act on a particle of mass 2 kg. Find its acceleration.

Resultant $=(6\mathbf{i}+4\mathbf{j})$ N, so $\mathbf{a}=(3\mathbf{i}+2\mathbf{j})$ m s^{-2}.

Constant acceleration in vector form

$\mathbf{v}=\mathbf{u}+\mathbf{a}t, \qquad \mathbf{s}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$

TRY IT

A particle starts at the origin with velocity $(2\mathbf{i}+\mathbf{j})$ m s^{-1} and acceleration $(\mathbf{i}-2\mathbf{j})$ m s^{-2}. Find its position after 4 s.

$\mathbf{s}=4(2\mathbf{i}+\mathbf{j})+\frac{1}{2}(16)(\mathbf{i}-2\mathbf{j})=(8\mathbf{i}+4\mathbf{j})+(8\mathbf{i}-16\mathbf{j})=(16\mathbf{i}-12\mathbf{j})$ m

Position vectors

A particle moving with constant velocity $\mathbf{v}$ from position $\mathbf{r}_0$ has position $\mathbf{r}=\mathbf{r}_0+\mathbf{v}t$. Two particles meet when their position vectors are equal at the same time t, so equate the $\mathbf{i}$ components and the $\mathbf{j}$ components.

NOTE

Equating components gives two equations. Use one to find t and check it in the other. If they disagree, the particles do not meet.

Moments

The **moment** of a force about a point is the force multiplied by the perpendicular distance from the point to the line of action of the force.

$$\text{moment}=F\times d \quad (\text{units N m})$$

A moment turns clockwise or anticlockwise. Choose one as positive and keep to it.

Equilibrium

A body is in equilibrium when the resultant force is zero **and** the total moment about any point is zero. In a problem about a rigid body at rest, take moments about a point where an unknown force acts, so that force drops out.

Note Academy

For equilibrium, clockwise moments equal anticlockwise moments about the same point.

TRY IT

A uniform plank AB of length 4 m and mass 20 kg rests horizontally on supports at A and at C, where $AC = 3$ m. Find the reaction at C.

Take moments about A. The weight acts at the middle, 2 m from A: $20g \times 2 = R_C \times 3$, so $R_C = \frac{40g}{3} = 131$ N (using $g = 9.8$).

TRY IT

A load of 5 kg is now placed at B. Find the reaction at C.

Moments about A: $20g \times 2 + 5g \times 4 = R_C \times 3$, so $R_C = \frac{60g}{3} = 20g = 196$ N.

A plank about to tilt

If a plank is on the point of tilting about a support, the reaction at the other support is zero. This gives the limiting position or the largest load.

NOTE

"On the point of tilting about C" means the reaction at the other support is zero. Take moments about C.

Non-uniform bodies

For a non-uniform rod, the weight acts at the centre of mass, which is not at the middle. Use the given position or the position you are asked to find.

TRY IT

A rod of mass 6 kg and length 3 m rests horizontally on supports at each end. Its centre of mass is 1 m from the left end A. Find the reaction at A.

Take moments about the right end B: $R_A \times 3 = 6g \times 2$, so $R_A = 4g = 39.2$ N.

Note Academy