Logarithms and exponentials

Logarithms and indices, the laws of logarithms, e to the x and ln x, equations with the unknown in a power, and turning a relationship into a straight line.

Logarithmic and exponential functions

Logarithms and indices

A logarithm is a power. These two statements say the same thing:

$$y = a^x \Leftrightarrow x = \log_a y$$

So $\log_2 8 = 3$ because $2^3 = 8$. Two values follow from the definition: $\log_a a = 1$ and $\log_a 1 = 0$.

The laws of logarithms

Law	Example
$\log_a(xy) = \log_a x + \log_a y$	$\log 6 = \log 2 + \log 3$
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log 5 = \log 10 - \log 2$
$\log_a(x^n) = n \log_a x$	$\log 8 = 3 \log 2$

TRY IT

Solve $\log_3 x + \log_3(x - 2) = 1$.

$\log_3[x(x-2)] = 1 \quad \Rightarrow \quad x(x-2) = 3 \quad \Rightarrow \quad (x-3)(x+1) = 0$

$x = -1$ would need the log of a negative number, so $x = 3$ only.

NOTE

There is no law for $\log(x + y)$. Always check the solutions: you cannot take the logarithm of zero or of a negative number.

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e^x and $\ln x$

The number $e \approx 2.718$ is the base for which $y = e^x$ has gradient equal to y at every point. The **natural logarithm** $\ln x$ means $\log_e x$.

e^x and $\ln x$ are inverse functions: $e^{\ln x} = x$ and $\ln(e^x) = x$. Their graphs are reflections in $y = x$.

Graph	Domain	Range	Passes through	Asymptote
$y = e^x$	All real x	$y > 0$	$(0, 1)$	$y = 0$
$y = \ln x$	$x > 0$	All real y	$(1, 0)$	$x = 0$

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Solve $e^{2x} = 7$, and $\ln(3x - 1) = 2$.

$2x = \ln 7$, so $x = 0.973$. For the second, $3x - 1 = e^2$, so $x = \dfrac{e^2 + 1}{3} = 2.80$.

The unknown in a power

Take logarithms of both sides to bring the power down.

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Solve $3^{2x+1} = 5^x$.

$(2x + 1)\ln 3 = x \ln 5 \quad \Rightarrow \quad x(2\ln 3 - \ln 5) = -\ln 3 \quad \Rightarrow \quad x = \frac{-\ln 3}{2\ln 3 - \ln 5} = -1.87$

Some equations are quadratic in a power.

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Solve $3^{2x} - 4(3^x) + 3 = 0$.

Let $u = 3^x$: $u^2 - 4u + 3 = 0$, so $u = 1$ or $u = 3$. Then $3^x = 1$ gives $x = 0$, and $3^x = 3$ gives $x = 1$.

Inequalities with the unknown in a power

The working is the same, but dividing by a negative logarithm reverses the inequality.

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Find the least integer n for which $0.8^n < 0.1$.

$n \ln 0.8 < \ln 0.1 \quad \Rightarrow \quad n > \frac{\ln 0.1}{\ln 0.8} = 10.3$

The sign reversed because $\ln 0.8$ is negative. So $n = 11$.

Turning a relationship into a straight line

Taking logarithms turns a curve into a straight line of the form $Y = mX + c$. The constants come from the gradient and intercept.

Relationship	Taking logs gives	Plot	Gradient	Intercept
$y = ax^n$	$\ln y = n \ln x + \ln a$	$\ln y$ against $\ln x$	n	$\ln a$
$y = ab^x$	$\ln y = x \ln b + \ln a$	$\ln y$ against x	$\ln b$	$\ln a$

TRY IT

$y = ax^n$. A graph of $\ln y$ against $\ln x$ is a straight line through $(1, 3)$ and $(3, 7)$. Find a and n.

The gradient is $n = \frac{7-3}{3-1} = 2$. The line is $\ln y = 2 \ln x + 1$, so $\ln a = 1$ and $a = \mathrm{e} = 2.72$.

NOTE

The intercept is the value of $\ln a$, so remember the last step of finding a itself.

NOTE

Read the axes. If they are labelled $\ln y$ and x, the relationship is the second kind, $y = ab^x$.

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