Integration

Integration as reverse differentiation, finding the constant, definite integrals, areas under and between curves, and volumes of revolution.

Integration

Reverse differentiation

Integration undoes differentiation. Add 1 to the power and divide by the new power.

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c \quad \textbf{for any rational } n \neq -1.$$

The constant c appears because any constant differentiates to zero. Rewrite roots and fractions as powers first, exactly as for differentiation.

TRY IT

Find $\int \left(6x^2 - \frac{4}{x^3} + \sqrt{x}\right) \mathrm{d}x.$

$\int \left(6x^2 - 4x^{-3} + x^{\frac{1}{2}}\right) \mathrm{d}x = 2x^3 + 2x^{-2} + \frac{2}{3}x^{\frac{3}{2}} + c$

Integrating $(ax + b)^n$

$$\int (ax+b)^n \,\mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad \textbf{for } n \neq -1.$$

The extra division by a undoes the factor the chain rule would bring out.

TRY IT

Find $\int (2x+1)^{\frac{1}{2}} \,\mathrm{d}x.$

$\frac{(2x+1)^{\frac{3}{2}}}{2\times\frac{3}{2}} + c = \frac{1}{3}(2x+1)^{\frac{3}{2}} + c$

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NOTE

Differentiating your answer is a quick check. It only works for a linear inside: $(x^2+1)^3$ cannot be integrated this way, and needs expanding first.

Finding the constant

If you know a point on the curve, substitute it to find c.

TRY IT

A curve has $\frac{dy}{dx} = 6x^2 - 4x$ and passes through $(1, 5)$. Find its equation.

$y = 2x^3 - 2x^2 + c$. At $(1, 5)$: $5 = 2 - 2 + c$, so $c = 5$ and $y = 2x^3 - 2x^2 + 5$.

Definite integrals

$\int_a^b f'(x)\,dx = f(b) - f(a)$. **No constant is needed.**

TRY IT

Evaluate $\int_0^4 (2x+1)^{\frac{1}{2}}\,dx$.

$\left[\frac{1}{3}(2x+1)^{\frac{3}{2}}\right]_0^4 = \frac{1}{3}(9^{\frac{3}{2}} - 1^{\frac{3}{2}}) = \frac{1}{3}(27-1) = \frac{26}{3}$

Area under a curve

The area between a curve, the x-axis and the lines $x = a$ and $x = b$ is $\int_a^b y\,dx$. Using $\int_c^d x\,dy$ gives the area between a curve and the y-axis.

TRY IT

Find the area under $y = x^2 + 1$ from $x = 1$ to $x = 3$.

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$$\int_1^3 (x^2+1)\,dx = \left[\tfrac{x^3}{3} + x\right]_1^3 = (9+3) - \left(\tfrac{1}{3}+1\right) = \tfrac{32}{3}$$

A region below the x-axis gives a negative integral. If the curve crosses the axis, integrate each part separately and add the sizes.

Area between two curves

$$\text{Area} = \int_a^b (\text{upper} - \text{lower})\,dx$$

This works wherever the region lies relative to the axis.

TRY IT

Find the area enclosed by $y = 4x - x^2$ and $y = x$.

They meet where $4x - x^2 = x$, so $x = 0$ or $x = 3$. Between these, the curve is above the line:

$$\int_0^3 (3x - x^2)\,dx = \left[\tfrac{3x^2}{2} - \tfrac{x^3}{3}\right]_0^3 = \tfrac{27}{2} - 9 = \tfrac{9}{2}$$

Volumes of revolution

Rotating a region through 360° about an axis gives a solid. Each thin slice is a disc of radius y (or x).

About the x-axis: $V = \pi \int_a^b y^2\,dx$. About the y-axis: $V = \pi \int_c^d x^2\,dy$.

TRY IT

The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is rotated about the x-axis. Find the volume.

$$V = \pi \int_0^4 x\,dx = \pi \left[\tfrac{x^2}{2}\right]_0^4 = 8\pi$$

TRY IT

The region between $y = x^2$, the y-axis, $y = 1$ and $y = 4$ is rotated about the y-axis. Find the volume.

$x^2 = y$, so

Note Academy

$$V = \pi \int_1^4 y \, dy = \pi \left[\frac{y^2}{2}\right]_1^4 = \pi \left(8 - \frac{1}{2}\right) = \frac{15\pi}{2}$$

NOTE

For a region between two curves, use $\pi \int (y_1^2 - y_2^2) \, dx$. It is the difference of the squares, not the square of the difference.

NOTE

A volume rotated about the y-axis needs limits in y and everything in terms of y. Mixing dx with y-limits is a common loss of marks.

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