Further trigonometry

Secant, cosecant and cotangent, the identities that use them, the addition and double angle formulae, and writing a sin x + b cos x as a single wave.

Trigonometry

Secant, cosecant and cotangent

$\sec\theta = \frac{1}{\cos\theta}$ $\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

Graph	Range	Asymptotes
$y = \sec x$	$y \le -1$ or $y \ge 1$	Where $\cos x = 0$: 90°, 270°
$y = \operatorname{cosec} x$	$y \le -1$ or $y \ge 1$	Where $\sin x = 0$: 0°, 180°, 360°
$y = \cot x$	All real values	Where $\sin x = 0$: 0°, 180°, 360°

Two more identities

Dividing $\sin^2\theta + \cos^2\theta \equiv 1$ by $\cos^2\theta$ and by $\sin^2\theta$ gives:

$\sec^2\theta \equiv 1 + \tan^2\theta$ **and** $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$

TRY IT

Solve $2\sec^2 x = 5\tan x$ for $0° \le x \le 360°$.

$2(1 + \tan^2 x) = 5\tan x \quad \Rightarrow \quad 2\tan^2 x - 5\tan x + 2 = 0 \quad \Rightarrow \quad (2\tan x - 1)(\tan x - 2) = 0$

$\tan x = \frac{1}{2}$ gives $x = 26.6°$ or $206.6°$. $\tan x = 2$ gives $x = 63.4°$ or $243.4°$.

The addition formulae

$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$

Note Academy

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

The signs flip in the cosine formula and the bottom of the tangent formula. These are on the formula list, but you must be able to use them quickly.

TRY IT

Find the exact value of $\sin 75°$.

$\sin(45° + 30°) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

TRY IT

Solve $\sin(x + 30°) = 2\cos x$ for $0° \leq x \leq 180°$.

$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2\cos x \quad \Rightarrow \quad \frac{\sqrt{3}}{2}\sin x = \frac{3}{2}\cos x \quad \Rightarrow \quad \tan x = \sqrt{3}$

So $x = 60°$.

The double angle formulae

Putting $B = A$ in the addition formulae gives:

$$\sin 2A \equiv 2\sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Choose the form of $\cos 2A$ that leaves only one ratio in the equation.

TRY IT

Note Academy

Solve $\cos 2x + 3\sin x = 2$ for $0° \le x \le 360°$.

$1 - 2\sin^2 x + 3\sin x = 2 \quad \Rightarrow \quad 2\sin^2 x - 3\sin x + 1 = 0 \quad \Rightarrow \quad (2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ gives $30°$ or $150°$. $\sin x = 1$ gives $90°$.

TRY IT

Solve $\sin 2x = \cos x$ for $0° \le x \le 360°$.

$2\sin x\cos x - \cos x = 0$, so $\cos x(2\sin x - 1) = 0$. Then $\cos x = 0$ gives $90°$ or $270°$, and $\sin x = \frac{1}{2}$ gives $30°$ or $150°$.

TRY IT

Prove that $\dfrac{1 - \cos 2x}{\sin 2x} \equiv \tan x$.

$\frac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$

Writing $a\sin\theta + b\cos\theta$ as a single wave

Any sum of a sine and a cosine of the same angle is one wave, written as $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$, with $R > 0$ and α acute.

$R = \sqrt{a^2 + b^2}$, and α comes from expanding and matching coefficients.

TRY IT

Express $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$.

$R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$. Matching gives $R\cos\alpha = 3$ and $R\sin\alpha = 4$.

So $R = 5$ and $\tan\alpha = \dfrac{4}{3}$, giving $\alpha = 53.13°$:

$3\sin x + 4\cos x = 5\sin(x + 53.13°)$

Using the single wave form

Solving equations. Solve $3\sin x + 4\cos x = 2$ for $0° \le x \le 360°$.

Note Academy

$\sin(x + 53.13°) = 0.4$, in the interval $53.13° \le x + 53.13° \le 413.13°$. The principal value is $23.58°$, which is outside the interval. The values inside are $156.42°$ and $383.58°$, so $x = 103.3°$ or $x = 330.4°$.

Greatest and least values. $5\sin(x + 53.13°)$ runs from -5 to 5. The greatest value is reached when $x + 53.13° = 90°$, so at $x = 36.87°$.

TRY IT

Find the greatest value of $\dfrac{7}{8 + 3\sin x + 4\cos x}$.

The denominator is $8 + 5\sin(x + 53.13°)$, whose least value is 3. So the greatest value of the fraction is $\dfrac{7}{3}$.

NOTE

Use the sign in the form the question asks for: $a\sin\theta - b\cos\theta$ goes to $R\sin(\theta - \alpha)$, and $a\cos\theta + b\sin\theta$ goes to $R\cos(\theta - \alpha)$. Keep the unrounded α in your calculator for the later steps.

Note Academy