Further integration

Standard integrals, using trigonometric identities, and the trapezium rule. It also adds partial fractions, integrals of f'(x)/f(x), integration by parts and substitution.

Integration

Standard integrals

Each of these is a reverse derivative. The divisor a undoes the chain rule, and works only when the inside is linear.

$f(x)$	$\int f(x)\,dx$
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert+c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)+c$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b)+c$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+c$

TRY IT

Find $\int \left(e^{2x+1}+\frac{1}{3x+2}+\sin 2x+\sec^2 3x\right)dx$.

$\frac{1}{2}e^{2x+1}+\frac{1}{3}\ln\lvert 3x+2\rvert-\frac{1}{2}\cos 2x+\frac{1}{3}\tan 3x+c$

TRY IT

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Evaluate $\int_0^1 \frac{1}{2x+1}\,\mathrm{d}x$.

$[\frac{1}{2}\ln(2x+1)]_0^1 = \frac{1}{2}\ln 3 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 3$

Using trigonometric identities

If a function cannot be integrated as it stands, change it with an identity.

Integrand	Rewrite as
$\cos^2 x$	$\frac{1}{2}(1+\cos 2x)$
$\sin^2 x$	$\frac{1}{2}(1-\cos 2x)$
$\tan^2 x$	$\sec^2 x - 1$
$\sin x \cos x$	$\frac{1}{2}\sin 2x$

TRY IT

Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x\,\mathrm{d}x$.

$\int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 2x)\,\mathrm{d}x = [\frac{x}{2} + \frac{\sin 2x}{4}]_0^{\frac{\pi}{4}} = \frac{\pi}{8} + \frac{1}{4}$

The trapezium rule

The trapezium rule estimates a definite integral by splitting the area into strips of equal width h and treating each as a trapezium.

$$\int_a^b y\,\mathrm{d}x \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$, **where** $h = \frac{b-a}{n}$ **for** n **intervals.**

TRY IT

Use 2 intervals to estimate $\int_0^2 \sqrt{1+x^2}\,\mathrm{d}x$.

$h = 1$, and the y-values are 1, $\sqrt{2}$ and $\sqrt{5}$:

$\frac{1}{2}[1 + 2\sqrt{2} + \sqrt{5}] = 3.03$

Note Academy

NOTE

n intervals means $n + 1$ values of y. If the curve bends upward (concave up), the trapezia sit above it, so the rule gives an **over-estimate**. If it bends downward, it gives an **under-estimate**.

Further methods

Choosing a method

The integrand looks like	Method
$\frac{1}{x^2 + a^2}$	Standard result with $\tan^{-1}$
A rational function with a factorised bottom	Partial fractions
$\frac{kf'(x)}{f(x)}$	Natural logarithm
A product such as xe^x or $x\cos x$, or $\ln x$ alone	Integration by parts
A substitution is given in the question	Substitution

Inverse tangent and partial fractions

TRY IT

Find $\int \frac{1}{x^2 + 9}\,dx$.

$\frac{1}{3}\tan^{-1}\frac{x}{3} + c$

Split a rational function first, then integrate each part.

TRY IT

Find $\int \frac{5x + 1}{(x + 1)(x - 1)}\,dx$.

The partial fractions are $\frac{2}{x + 1} + \frac{3}{x - 1}$, so the integral is $2\ln|x + 1| + 3\ln|x - 1| + c$.

A squared bracket gives a power, not a logarithm: $\int \frac{2}{(x + 1)^2}\,dx = -\frac{2}{x + 1} + c$.

Note Academy

Integrals of the form $\dfrac{kf'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Look for a top that is a multiple of the derivative of the bottom.

TRY IT

Find $\int \dfrac{x}{x^2+1}\,dx$ and $\int \tan x\,dx$.

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + c$

$\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Choose u to be the part that gets simpler when differentiated. The exception is a logarithm: make $u = \ln x$, since you cannot integrate it directly.

TRY IT

Find $\int xe^{2x}\,dx$.

$u = x$ and $\dfrac{dv}{dx} = e^{2x}$, so $v = \frac{1}{2}e^{2x}$:

$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

TRY IT

Evaluate $\int_1^e \ln x\,dx$.

$u = \ln x$ and $\dfrac{dv}{dx} = 1$, so $v = x$:

Note Academy

$$[x \ln x]_1^e - \int_1^e 1 \, dx = e - (e - 1) = 1$$

Substitution

Replace x everywhere, including dx and the limits.

> **TRY IT**
>
> Use $u = x + 1$ to evaluate $\int_0^3 x\sqrt{x+1} \, dx$.

$du = dx$ and $x = u - 1$. The limits become $u = 1$ and $u = 4$:

$$\int_1^4 (u-1)u^{\frac{1}{2}} \, du = \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4 = \frac{2}{5}(31) - \frac{2}{3}(7) = \frac{116}{15}$$

> **NOTE**
>
> Change the limits and you never need to go back to x. For an indefinite integral, there are no limits, so finish in terms of x.

Note Academy