Further differentiation

Derivatives of exponentials, logarithms and trigonometric functions, the product and quotient rules, and curves defined parametrically or implicitly. It also adds the inverse tangent.

Differentiation

Standard derivatives

Angles must be in radians for these results.

y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

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Composites

The chain rule gives these general results:

y	$\frac{dy}{dx}$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$
$\sin f(x)$	$f'(x)\cos f(x)$
$\cos f(x)$	$-f'(x)\sin f(x)$
$\tan f(x)$	$f'(x)\sec^2 f(x)$

TRY IT

Differentiate e^{3x^2}, $\ln(3x^2+1)$, $\tan 2x$ and $\sin^2 x$.

$6x\,e^{3x^2}$ $\quad \frac{6x}{3x^2+1}$ $\quad 2\sec^2 2x$ $\quad 2\sin x\cos x = \sin 2x$

$\sin^2 x$ means $(\sin x)^2$, so use the power rule first and multiply by the derivative of $\sin x$.

The product and quotient rules

Product: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

TRY IT

Differentiate $y = x^2e^{2x}$.

$\frac{dy}{dx} = x^2 \cdot 2e^{2x} + e^{2x} \cdot 2x = 2xe^{2x}(x+1)$

TRY IT

Differentiate $y = \frac{\ln x}{x}$.

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$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$

The quotient rule also gives these results, which are worth recognising:

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Stationary points with these functions

Set the derivative to zero and solve. A positive factor like e^{-x} can never be zero, so divide it out safely.

TRY IT

Find the stationary point of $y = xe^{-x}$.

$\frac{dy}{dx} = e^{-x} - xe^{-x} = (1 - x)e^{-x}$. Since $e^{-x} > 0$, the only solution is $x = 1$, at the point $(1, e^{-1})$.

The gradient goes from positive to negative through $x = 1$, so this is a maximum.

Parametric curves

A curve can be described by writing x and y in terms of a third variable, the **parameter** t.

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

TRY IT

A curve has $x = t^2 + 1$ and $y = 2t^3$. Find the tangent at $t = 1$.

$\frac{dy}{dx} = \frac{6t^2}{2t} = 3t$, which is 3 at $t = 1$. The point is $(2, 2)$, so the tangent is $y - 2 = 3(x - 2)$, or $y = 3x - 4$.

Implicit differentiation

When y is mixed in with x, differentiate every term with respect to x. Any term in y picks up a factor of $\frac{dy}{dx}$, and a product of x and y needs the product rule.

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Term	Derivative with respect to x
y^2	$2y\frac{dy}{dx}$
xy	$x\frac{dy}{dx}+y$
$\sin y$	$\cos y\frac{dy}{dx}$

TRY IT

Find the gradient of $x^2+xy+y^2=7$ at the point $(1,2)$.

$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{2x+y}{x+2y}$

At $(1,2)$ the gradient is $-\frac{4}{5}$.

NOTE

Stationary points of an implicit curve come from setting the top of the fraction to zero, then solving that with the curve's own equation.

The inverse tangent

$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$ $\frac{d}{dx}(\tan^{-1}2x)=\frac{2}{1+4x^2}$

Note Academy