Functions

Domain and range, composite functions, one-one functions and their inverses, and transformations of the graph of y = f(x).

Functions

The language of functions

A **function** takes each input to exactly one output.

Term	Meaning
Domain	The set of allowed inputs
Range	The set of outputs the function actually gives
One-one	Each output comes from only one input
Many-one	Some output comes from more than one input

$f(x) = x^2$ for all real x is many-one, because $f(2) = f(-2)$. With the domain $x \geq 0$ it becomes one-one.

Finding the range

Sketch the graph over the given domain, or complete the square for a quadratic, and read off the y-values.

> **TRY IT**
>
> Find the range of $f(x) = x^2 - 4x + 7$ for $x \in \mathbb{R}$.

$f(x) = (x - 2)^2 + 3$, and the bracket is never negative, so $f(x) \geq 3$.

> **TRY IT**
>
> Find the range of $f(x) = x^2 - 4x + 7$ for $3 \leq x \leq 5$.

On this domain the function is increasing, since the turning point at $x = 2$ is outside it. $f(3) = 4$ and $f(5) = 12$, so $4 \leq f(x) \leq 12$.

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NOTE

Check where the turning point lies. On $0 \le x \le 5$ the least value is at $x = 2$, not at either end, so the range is $3 \le \mathrm{f}(x) \le 12$.

Composite functions

$\mathrm{fg}(x)$ means $\mathrm{f}(\mathrm{g}(x))$: apply g first, then f.

TRY IT

Given $\mathrm{f}(x) = 2x + 3$ and $\mathrm{g}(x) = x^2$, find $\mathrm{fg}(x)$ and $\mathrm{gf}(x)$.

$\mathrm{fg}(x) = 2x^2 + 3 \qquad \mathrm{gf}(x) = (2x + 3)^2$

They are different. Order matters.

fg exists only if every output of g is an allowed input of f: the range of g must lie within the domain of f.

Inverse functions

Only a one-one function has an inverse. The inverse undoes the function, so $\mathrm{f}^{-1}\mathrm{f}(x) = x$.

To find it: write $y = \mathrm{f}(x)$, make x the subject, then swap the letters.

TRY IT

Find $\mathrm{f}^{-1}(x)$ for $\mathrm{f}(x) = (x - 1)^2 + 2, x \ge 1$.

$y = (x - 1)^2 + 2 \quad \Rightarrow \quad x - 1 = \pm\sqrt{y - 2}$

The domain $x \ge 1$ means $x - 1 \ge 0$, so take the positive root. Then

$\mathrm{f}^{-1}(x) = 1 + \sqrt{x - 2}, \quad x \ge 2$

TRY IT

Find the inverse of $\mathrm{f}(x) = \dfrac{2x + 1}{x - 3}, x \neq 3$.

$y(x - 3) = 2x + 1 \quad \Rightarrow \quad xy - 2x = 1 + 3y \quad \Rightarrow \quad x = \frac{3y+1}{y-2}$

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So $\mathrm{f}^{-1}(x) = \dfrac{3x+1}{x-2}, x \neq 2$.

> **The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f. Always state the domain of the inverse.**

Graphs of a function and its inverse

The graph of $y = \mathrm{f}^{-1}(x)$ is the reflection of $y = \mathrm{f}(x)$ in the line $y = x$. Any point where the two meet on that line satisfies $\mathrm{f}(x) = x$.

Making a function one-one

A quadratic is one-one only on one side of its turning point.

> **TRY IT**
>
> $\mathrm{f}(x) = x^2 - 4x + 1$ for $x \geq k$. Find the least value of k for which f has an inverse.

The turning point is at $x = 2$, since $\mathrm{f}(x) = (x-2)^2 - 3$. So the least value is $k = 2$.

Transformations of graphs

Equation	Transformation of $y = \mathrm{f}(x)$
$y = \mathrm{f}(x) + a$	Translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$: up by a
$y = \mathrm{f}(x + a)$	Translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$: left by a
$y = a\mathrm{f}(x)$	Stretch parallel to the y-axis, scale factor a
$y = \mathrm{f}(ax)$	Stretch parallel to the x-axis, scale factor $\dfrac{1}{a}$
$y = -\mathrm{f}(x)$	Reflection in the x-axis
$y = \mathrm{f}(-x)$	Reflection in the y-axis

> **Changes inside the bracket act on x and do the opposite of what they look like: $\mathrm{f}(x-2)$ moves**

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right, and $f(2x)$ squashes by a factor of $\frac{1}{2}$.

NOTE

Use the exam's words: **translation** with a vector, **stretch** with a direction and a scale factor, **reflection** with a line. "Shift" and "moves" lose marks.

Combining transformations

Apply the changes inside the bracket first, then those outside. Outside the bracket, a multiplication comes before an addition, just as in arithmetic.

TRY IT

Describe how $y = x^2$ becomes $y = 3(x-2)^2 + 1$.

1. Translation by $\begin{pmatrix}2\\0\end{pmatrix}$, giving $y = (x-2)^2$.
2. Stretch parallel to the y-axis, scale factor 3, giving $y = 3(x-2)^2$.
3. Translation by $\begin{pmatrix}0\\1\end{pmatrix}$.

TRY IT

A curve $y = f(x)$ is reflected in the x-axis and then translated by $\begin{pmatrix}0\\4\end{pmatrix}$. Write the new equation.

Reflecting gives $y = -f(x)$. Translating gives $y = 4 - f(x)$. Doing them in the other order gives $y = -f(x) - 4$, which is different.

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