Discrete random variables

Probability distribution tables, expectation and variance, the binomial distribution, and the geometric distribution.

Discrete random variables

Probability distributions

A **discrete random variable** takes separate values, each with a probability. A probability distribution lists them, and the probabilities add to 1.

TRY IT

$\mathrm{P}(X = x) = kx$ for $x = 1, 2, 3, 4$. Find k.

$k + 2k + 3k + 4k = 1$, so $k = 0.1$.

To draw up a table from a situation, list every possible value of X first, then work out each probability. Check that they total 1.

Expectation and variance

$\mathrm{E}(X) = \sum xp$ **and** $\mathrm{Var}(X) = \sum x^2 p - [\mathrm{E}(X)]^2$

TRY IT

Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

x	1	2	3
$\mathrm{P}(X = x)$	0.2	0.5	0.3

$\mathrm{E}(X) = 0.2 + 1.0 + 0.9 = 2.1$

$\sum x^2 p = 0.2 + 2.0 + 2.7 = 4.9$, so $\mathrm{Var}(X) = 4.9 - 2.1^2 = 0.49$.

NOTE

Note Academy

Do not divide by the number of values, since the probabilities are already the weights. Do not forget to square $\mathrm{E}(X)$.

The binomial distribution

$X \sim \mathrm{B}(n, p)$ counts the number of successes in n trials. It is suitable when:

- There is a fixed number of trials, n.
- Each trial has two outcomes, success and failure.
- The trials are independent.
- The probability p of success is constant.

$\mathrm{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$, $\mathrm{E}(X) = np$ **and** $\mathrm{Var}(X) = np(1-p)$

TRY IT

$X \sim \mathrm{B}(10, 0.3)$. Find $\mathrm{P}(X = 2)$ and $\mathrm{P}(X \geq 1)$.

$\mathrm{P}(X = 2) = \binom{10}{2}(0.3)^2(0.7)^8 = 45 \times 0.09 \times 0.05765 = 0.233$

$\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0) = 1 - 0.7^{10} = 0.972$

Inequalities

Translate the words carefully:

Words	Means	For $\mathrm{B}(10, p)$
At most 2	$X \leq 2$	Add $\mathrm{P}(0) + \mathrm{P}(1) + \mathrm{P}(2)$
Fewer than 2	$X \leq 1$	Add $\mathrm{P}(0) + \mathrm{P}(1)$
At least 2	$X \geq 2$	$1 - \mathrm{P}(0) - \mathrm{P}(1)$
More than 2	$X \geq 3$	$1 - \mathrm{P}(0) - \mathrm{P}(1) - \mathrm{P}(2)$

Finding the number of trials

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An event has probability 0.3 on each trial. Find the least n for which the probability of at least one success is greater than 0.95.

$1 - 0.7^n > 0.95 \quad \Rightarrow \quad 0.7^n < 0.05 \quad \Rightarrow \quad n > \frac{\ln 0.05}{\ln 0.7} = 8.4$

So $n = 9$. The inequality reverses because $\ln 0.7$ is negative.

The geometric distribution

$X \sim \text{Geo}(p)$ counts the number of trials up to and including the first success. There is no upper limit. The trials are independent, with constant probability of success, as for the binomial, but with no fixed number of them.

$\text{P}(X = r) = (1-p)^{r-1}p$ **and** $\text{E}(X) = \frac{1}{p}$

Getting the first success on trial r needs $r - 1$ failures, then one success.

These two results come from the same idea:

$\text{P}(X > r) = (1-p)^r \qquad \text{P}(X \leq r) = 1 - (1-p)^r$

$\text{P}(X > r)$ is the probability that the first r trials are all failures.

TRY IT

A fair die is thrown until a 6 appears. Find the probability that the first 6 comes on the 3rd throw, the probability that more than 4 throws are needed, and the expected number of throws.

$\text{P}(X = 3) = (\frac{5}{6})^2 \frac{1}{6} = \frac{25}{216} = 0.116$

$\text{P}(X > 4) = (\frac{5}{6})^4 = 0.482 \qquad \text{E}(X) = 6$

Which distribution?

Question	Distribution
How many successes in a fixed number of trials?	Binomial
How many trials until the first success?	Geometric

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Look at what is being counted. "In the next 10 games, how many wins" is binomial. "How many games up to the first win" is geometric.

NOTE

When asked why a binomial model might not suit, name a condition and link it to the context: "The trials may not be independent, since people in one family may share the same opinion."

When a question combines distributions

A common type of question has two levels: first, the probability for one group, then a binomial on the number of groups.

TRY IT

A bag of 10 seeds counts as "good" if at least 9 grow, and each seed grows with probability 0.9. Find the probability that exactly 2 of 3 bags are good.

For one bag: $\mathrm{P}(\text{good}) = 10(0.9)^9(0.1) + (0.9)^{10} = 0.3874 + 0.3487 = 0.7361.$

For the bags, with $\mathrm{B}(3, 0.7361)$:

$\binom{3}{2}(0.7361)^2(0.2639) = 0.429$

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