Differentiation

The gradient as a limit, the power rule and chain rule, tangents and normals, increasing and decreasing functions, rates of change, and stationary points.

Differentiation

The gradient of a curve

The gradient of a curve at a point is the gradient of its tangent there. It is the limit of the gradients of chords from that point, as the other end of the chord moves closer and closer.

Notation	Meaning
$\frac{dy}{dx}$ or $f'(x)$	First derivative: the gradient
$\frac{d^2y}{dx^2}$ or $f''(x)$	Second derivative: the rate of change of the gradient

The power rule

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, for any rational n. A constant multiple stays, and a constant on its own differentiates to 0.

Rewrite roots and fractions as powers first:

$\sqrt{x} = x^{\frac{1}{2}}$ $\quad \frac{3}{x^2} = 3x^{-2}$ $\quad \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

TRY IT

Differentiate $y = 4x^3 - \frac{3}{x^2} + 2\sqrt{x}$.

$y = 4x^3 - 3x^{-2} + 2x^{\frac{1}{2}}$, so

$\frac{dy}{dx} = 12x^2 + 6x^{-3} + x^{-\frac{1}{2}}$

Note Academy

The chain rule

For a function of a function, differentiate the outside, keep the inside, then multiply by the derivative of the inside.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. **So** $\frac{d}{dx}[(ax+b)^n] = an(ax+b)^{n-1}$.

TRY IT

Differentiate $y = (3x+1)^4$ and $y = \sqrt{2x+1}$.

$\frac{d}{dx}(3x+1)^4 = 4(3x+1)^3 \times 3 = 12(3x+1)^3$

$\frac{d}{dx}(2x+1)^{\frac{1}{2}} = \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2 = (2x+1)^{-\frac{1}{2}}$

Tangents and normals

The tangent has the gradient of the curve at that point. The normal is perpendicular to it.

TRY IT

Find the tangent and normal to $y = x^3 - 2x$ at $x = 2$.

At $x = 2$, $y = 4$ and $\frac{dy}{dx} = 3x^2 - 2 = 10$.

Tangent: $y - 4 = 10(x-2)$, so $y = 10x - 16$.

Normal: the gradient is $-\frac{1}{10}$, so $y - 4 = -\frac{1}{10}(x-2)$, which is $x + 10y = 42$.

Increasing and decreasing functions

Condition	The function is
$f'(x) > 0$	Increasing
$f'(x) < 0$	Decreasing

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Show that $f(x) = x^3 + 3x^2 + 6x$ is an increasing function.

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$f'(x) = 3x^2 + 6x + 6 = 3\left[(x+1)^2 + 1\right]$. The bracket is always at least 1, so $f'(x) > 0$ for all x.

> **NOTE**
>
> Completing the square is the usual way to show a quadratic derivative never changes sign. A function that is always increasing or always decreasing is one-one, so it has an inverse.

Rates of change

When quantities change with time, connect them with the chain rule.

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

> **TRY IT**
>
> The radius of a circle grows at 0.2 cm s^{-1}. Find the rate of increase of its area when the radius is 5 cm.

$A = \pi r^2$, so $\frac{dA}{dr} = 2\pi r = 10\pi$ when $r = 5$.

$\frac{dA}{dt} = 10\pi \times 0.2 = 2\pi \text{ cm}^2 \text{ s}^{-1}$

Stationary points

A **stationary point** is where $\frac{dy}{dx} = 0$: the tangent is horizontal.

Use the second derivative to decide which kind it is:

$\frac{d^2y}{dx^2}$	**Nature**
Positive	Minimum
Negative	Maximum
Zero	Inconclusive: check the gradient on each side

> **TRY IT**
>
> Find and classify the stationary points of $y = x^3 - 3x^2 - 9x + 5$.

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$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x-3)(x+1) = 0 \quad \Rightarrow \quad x = 3 \text{ or } x = -1$

The points are $(3, -22)$ and $(-1, 10)$. The second derivative is $6x - 6$:

- At $x = 3$ it is $12 > 0$, so $(3, -22)$ is a minimum.
- At $x = -1$ it is $-12 < 0$, so $(-1, 10)$ is a maximum.

Maximum and minimum problems

Write the quantity in terms of one variable, differentiate, set to zero, and confirm the nature.

TRY IT

A rectangle has perimeter 40 cm. Find the greatest possible area.

If one side is x, the other is $20 - x$, so $A = x(20 - x) = 20x - x^2$.

$\frac{dA}{dx} = 20 - 2x = 0$ gives $x = 10$. $\frac{d^2A}{dx^2} = -2 < 0$, so this is a maximum, and the greatest area is 100 cm².

NOTE

Questions like this often say "show that $A = \ldots$" first. Use the given result in the next part even if you could not show it.

Note Academy