Differential equations

Writing a rate of change as a differential equation, separating the variables, using an initial condition, and interpreting the solution in context.

Differential equations

Writing a differential equation

A **differential equation** connects a quantity to its rate of change. The word "proportional" brings a constant into the equation.

Statement	Equation
The population N grows at a rate proportional to N	$\frac{\mathrm{d}N}{\mathrm{d}t} = kN,\ k > 0$
The mass m decreases at a rate proportional to m	$\frac{\mathrm{d}m}{\mathrm{d}t} = -km,\ k > 0$
The temperature θ falls at a rate proportional to its excess over a room at 20°	$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20),\ k > 0$
The rate of change of y is inversely proportional to x	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{x}$

Separating the variables

If the right-hand side is a function of x multiplied by a function of y, gather each variable on its own side:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y) \quad \Rightarrow \quad \int \frac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \int \mathrm{f}(x)\,\mathrm{d}x$$

Only one constant of integration is needed, on one side.

TRY IT

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$ given that $y = 1$ when $x = 0$.

$$\int y^{-2}\,\mathrm{d}y = \int 2x\,\mathrm{d}x \quad \Rightarrow \quad -\tfrac{1}{y} = x^2 + c$$

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At $x = 0, y = 1$: $-1 = c$. So $-\frac{1}{y} = x^2 - 1$, which gives

$y = \frac{1}{1-x^2}$

General and particular solutions

The answer with c still in it is the **general solution**. Using an **initial condition** to find c gives the **particular solution**.

> **NOTE**
>
> Find c as soon as you have integrated. It keeps the rearranging simple, particularly when logarithms are involved.

Exponential growth and decay

> **TRY IT**
>
> Solve $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$ given that $N = N_0$ when $t = 0$.

$\int \frac{1}{N}\,\mathrm{d}N = \int k\,\mathrm{d}t \quad \Rightarrow \quad \ln N = kt + c$

Taking exponentials gives $N = \mathrm{e}^{kt+c} = A\mathrm{e}^{kt}$, where $A = \mathrm{e}^c$. At $t = 0$, $A = N_0$, so $N = N_0\mathrm{e}^{kt}$.

> **Removing a logarithm does not leave $+c$ on the end: e^{kt+c} is $\mathrm{e}^c \times \mathrm{e}^{kt}$, a constant multiple.**

A problem in context

> **TRY IT**
>
> A cup of tea at 100°C cools in a room at 20°C. Its temperature θ satisfies $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20)$, where t is in minutes. After 5 minutes, the temperature is 60°C. Find θ in terms of t, and what happens for large t.

Separate and integrate:

$\int \frac{1}{\theta-20}\,\mathrm{d}\theta = \int -k\,\mathrm{d}t \quad \Rightarrow \quad \ln(\theta - 20) = -kt + c$

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At $t = 0$, $\theta = 100$, so $c = \ln 80$. Then $\theta - 20 = 80\mathrm{e}^{-kt}$.

At $t = 5$, $\theta = 60$: $40 = 80\mathrm{e}^{-5k}$, so $k = \dfrac{\ln 2}{5} = 0.139$.

$\theta = 20 + 80\mathrm{e}^{-0.139t}$

As t grows, $\mathrm{e}^{-0.139t}$ approaches 0, so θ approaches 20°C: the tea cools to room temperature and never falls below it.

Combining with earlier methods

Differential equations are often used to test integration. Watch for these forms after separating:

Integral on one side	Method
$\dfrac{1}{y(y+1)}$	Partial fractions
$x\mathrm{e}^x$ or $x\cos x$	Integration by parts
$\cos^2 y$	Double angle formula
$\dfrac{x}{x^2+1}$	$\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ gives a logarithm
$\dfrac{1}{y^2+4}$	Inverse tangent

TRY IT

Solve $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + 1$ for $x > 0$, given that $y = 3$ when $x = 1$.

$\int \frac{1}{y+1}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x \quad \Rightarrow \quad \ln(y+1) = \ln x + c$

At $x = 1$, $y = 3$: $c = \ln 4$. So $\ln(y+1) = \ln(4x)$, and $y = 4x - 1$.

NOTE

Questions often end by asking what happens as t grows, or whether a limit is ever reached. Answer in the words of the context.

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