Algebra

The modulus function, polynomial division, the factor and remainder theorems. partial fractions and the binomial expansion for any rational power.

Modulus

What $|x|$ means

$|x|$ is the size of x, ignoring its sign: $|3| = 3$ and $|-3| = 3$. It is never negative.

The graph of $y = |ax + b|$ is the straight line $y = ax + b$ with any part below the x-axis reflected upwards, giving a V shape with its vertex on the axis at $x = -\frac{b}{a}$.

> $|a| = |b| \Leftrightarrow a^2 = b^2$, **and** $|x - a| < b \Leftrightarrow a - b < x < a + b$.

Solving modulus equations

Squaring both sides removes the modulus. It is safe when both sides are moduli.

> **TRY IT**
>
> Solve $|x - 1| = |2x + 3|$.

$(x-1)^2 = (2x+3)^2 \quad \Rightarrow \quad 3x^2 + 14x + 8 = 0 \quad \Rightarrow \quad (3x+2)(x+4) = 0$

So $x = -\frac{2}{3}$ or $x = -4$.

When one side has no modulus, sketch both graphs, or consider each case and check the answers.

> **TRY IT**
>
> Solve $|2x - 3| = x$.

$2x - 3 = x$ gives $x = 3$. $2x - 3 = -x$ gives $x = 1$. Both check, since the right side is positive in each case.

> **NOTE**

Note Academy

Squaring or splitting into cases can give false solutions when one side is not a modulus. Substitute each answer into the original equation.

Solving modulus inequalities

Find the critical values as for the equation, then use a sketch to pick the region.

> **TRY IT**
>
> Solve $|2x - 3| < x$.

The critical values are $x = 1$ and $x = 3$. The V is below the line between them, so $1 < x < 3$.

> **TRY IT**
>
> Solve $|x - 2| > |x + 4|$.

$(x-2)^2 > (x+4)^2 \quad \Rightarrow \quad -4x + 4 > 8x + 16 \quad \Rightarrow \quad x < -1$

Polynomials

Algebraic division

A polynomial $P(x)$ divided by a divisor gives a quotient and a remainder:

$$P(x) = \text{divisor} \times \text{quotient} + \text{remainder}$$

The remainder has a lower degree than the divisor: a number for a linear divisor, and a linear expression for a quadratic divisor.

> **TRY IT**
>
> Divide $2x^3 + 3x^2 - 5x + 4$ by $x + 2$.

Divide each leading term by x, multiply back, subtract and bring down the next term:

Next term of the quotient	Subtract	Leaves
$2x^2$	$2x^2(x+2) = 2x^3 + 4x^2$	$-x^2 - 5x$
$-x$	$-x(x+2) = -x^2 - 2x$	$-3x + 4$
-3	$-3(x+2) = -3x - 6$	10

Note Academy

The quotient is $2x^2 - x - 3$ and the remainder is 10.

NOTE

Comparing coefficients gets the same answer: write $\mathrm{P}(x) = (x+2)(ax^2+bx+c) + R$, expand, and match each power. A missing power in the polynomial has coefficient 0, so write it in.

The remainder and factor theorems

Remainder theorem: when $\mathrm{P}(x)$ is divided by $(ax-b)$, the remainder is $\mathrm{P}(\frac{b}{a})$.

Factor theorem: $(ax-b)$ is a factor of $\mathrm{P}(x)$ if and only if $\mathrm{P}(\frac{b}{a}) = 0$.

When a question has two unknowns, two conditions give simultaneous equations.

TRY IT

$\mathrm{P}(x) = x^3 + ax^2 + bx + 6$ has factor $(x-2)$ and remainder 12 on dividing by $(x+1)$. Find a and b.

$\mathrm{P}(2) = 8 + 4a + 2b + 6 = 0$, so $2a + b = -7$.

$\mathrm{P}(-1) = -1 + a - b + 6 = 12$, so $a - b = 7$.

Adding gives $3a = 0$, so $a = 0$ and $b = -7$.

Factorising a cubic

Find one factor by trying small values, divide, then factorise the quadratic.

TRY IT

Solve $x^3 - 2x^2 - 5x + 6 = 0$.

$\mathrm{P}(1) = 1 - 2 - 5 + 6 = 0$, so $(x-1)$ is a factor. Dividing gives $(x-1)(x^2 - x - 6) = (x-1)(x-3)(x+2)$.

The solutions are $x = 1$, $x = 3$ and $x = -2$.

Note Academy

Partial fractions

The forms to use

A fraction with factorised denominator can be split into simpler fractions, which are easier to integrate and to expand.

Denominator	Form
$(ax+b)(cx+d)(ex+f)$	$\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{ex+f}$
$(ax+b)(cx+d)^2$	$\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}$
$(ax+b)(cx^2+d)$	$\frac{A}{ax+b}+\frac{Bx+C}{cx^2+d}$

TRY IT

Express $\frac{5x+1}{(x+1)(x-1)}$ in partial fractions.

Write $5x+1=A(x-1)+B(x+1)$.

Put $x=1$: $6=2B$, so $B=3$. Put $x=-1$: $-4=-2A$, so $A=2$.

$\frac{2}{x+1}+\frac{3}{x-1}$

TRY IT

Express $\frac{x+5}{(x-1)(x+1)^2}$ in partial fractions.

Write $x+5=A(x+1)^2+B(x-1)(x+1)+C(x-1)$.

Put $x=1$: $6=4A$, so $A=\frac{3}{2}$. Put $x=-1$: $4=-2C$, so $C=-2$. Compare x^2 terms: $0=A+B$, so $B=-\frac{3}{2}$.

$\frac{3}{2(x-1)}-\frac{3}{2(x+1)}-\frac{2}{(x+1)^2}$

NOTE

Note Academy

If the degree of the top is not less than the degree of the bottom, divide first, so that a polynomial part comes out in front of the partial fractions.

Binomial expansion for any rational power

The expansion

When n is not a positive integer, the expansion never ends and only works for small x.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{ for } |x| < 1.$$

TRY IT

Expand $(1-2x)^{-3}$ up to the term in x^2, and state where it is valid.

Replace x with $-2x$ and put $n = -3$:

$$1 + (-3)(-2x) + \frac{(-3)(-4)}{2}(-2x)^2 = 1 + 6x + 24x^2$$

It is valid for $|2x| < 1$, so $|x| < \frac{1}{2}$.

When the first term is not 1

Take the first term out as a factor, including its power.

TRY IT

Expand $\sqrt{4+x}$ up to the term in x^2.

$$\sqrt{4+x} = 4^{\frac{1}{2}}\left(1+\tfrac{x}{4}\right)^{\frac{1}{2}} = 2\left[1 + \tfrac{1}{2}\cdot\tfrac{x}{4} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\tfrac{x}{4}\right)^2\right] = 2 + \tfrac{x}{4} - \tfrac{x^2}{64}$$

It is valid for $|x| < 4$.

Partial fractions are often used first, so that each part can be expanded separately and the results added.

Note Academy