Transformations and Vectors

Reflection, rotation, translation and enlargement of shapes, with the coordinate rules for each, and vector addition, magnitude, parallel vectors and collinear points.

Transformations

Reflection

The shape is flipped over a mirror line (like the x-axis, the y-axis or any line like $x = 2$ or $y = -1$). The perpendicular distance of a point from the line of reflection remains the same.

TRY IT

Reflect the shape over the line R (the line $x = 2$).

As demonstrated in the question above, you have to make sure to keep the perpendicular distance of the new shape from the line of reflection the same as the original shape.

TRY IT

Reflect shape A in $y = x$.

To make things easier, memorise the table below that shows how the coordinates change when you reflect a shape.

Line of reflection	Rule for coordinates	Example: $A(3, 2)$ becomes
x-axis	Change the sign of y: $(x, y) \to (x, -y)$	$(3, -2)$
y-axis	Change the sign of x: $(x, y) \to (-x, y)$	$(-3, 2)$
$y = x$	Swap x and y: $(x, y) \to (y, x)$	$(2, 3)$
$y = -x$	Swap x and y and change both signs: $(x, y) \to (-y, -x)$	$(-2, -3)$
$x = a$	The distance from the line stays the same, horizontally	in $x = 1$: $(-1, 2)$
$y = b$	The distance from the line stays the same, vertically	in $y = 4$: $(3, 6)$

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NOTE

For $x = a$ the new x-coordinate is $2a - x$, and for $y = b$ the new y-coordinate is $2b - y$. In the examples, $2(1) - 3 = -1$ and $2(4) - 2 = 6$.

Rotation

This is like turning a shape around a point, called the **centre of rotation**, by a certain angle and direction (clockwise or anticlockwise).

Rotation	Coordinate rule	Example: $A(3, 2)$ becomes
$90°$ clockwise ($-90°$)	$(x, y) \to (y, -x)$	$(2, -3)$
$90°$ anticlockwise ($+90°$)	$(x, y) \to (-y, x)$	$(-2, 3)$
$180°$	$(x, y) \to (-x, -y)$	$(-3, -2)$
$270°$ clockwise	Same as $90°$ anticlockwise	$(-2, 3)$
$270°$ anticlockwise	Same as $90°$ clockwise	$(2, -3)$
$360°$	No change	$(3, 2)$

These rules turn the shape about the **origin**. Memorise the table.

TRY IT

Rotate shape A by $90°$ in the clockwise direction around the point $(0, 1)$.

In this question, $(0, 1)$ is the centre of rotation. Each corner stays the same distance from $(0, 1)$ and turns a quarter turn clockwise around it. You can check a corner with the rule: move the centre to the origin (subtract $(0, 1)$), apply $(x, y) \to (y, -x)$, then move back (add $(0, 1)$). For the corner $(-3, 5)$:

$$(-3, 5) \to (-3, 4) \to (4, 3) \to (4, 4)$$

Translation

A translation is written like this: $\begin{pmatrix} a \\ b \end{pmatrix}$.

- a = how many steps right ($+$) or left ($-$).
- b = how many steps up ($+$) or down ($-$).

Note Academy

Translation vector	What it does	Example: $A(2, 3)$ becomes
$\binom{3}{2}$	Right 3, up 2	$(5, 5)$
$\binom{-4}{1}$	Left 4, up 1	$(-2, 4)$
$\binom{0}{-5}$	No horizontal movement, down 5	$(2, -2)$

TRY IT

Translate shape A by a translation of $\binom{4}{-1}$.

Every corner moves 4 right and 1 down.

Enlargement

This changes the size of the shape, but not the shape itself. It can grow or shrink based on the scale factor k.

Scale factor (k)	What happens	Example: $A(2, 3)$ becomes
2	Shape becomes twice as big	$(4, 6)$
0.5	Shape becomes half the size	$(1, 1.5)$
-1	Shape stays the same size but flips	$(-2, -3)$
-2	Shape doubles and flips across the centre	$(-4, -6)$

It should also be noted that the shape changes from the perspective of a **centre of enlargement**. The table above shows the change in coordinates if the centre of enlargement is the origin $(0, 0)$.

TRY IT

Enlarge the shape below by a scale factor of 2 with the origin as the centre of enlargement.

Draw a line from the origin through each corner, and go twice as far along it. The corners $(1, 3)$, $(1, 1)$ and $(2, 1)$ become $(2, 6)$, $(2, 2)$ and $(4, 2)$.

TRY IT

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Perform an enlargement of scale factor $-\frac{1}{2}$ on the shape below, with the centre of enlargement at $(-1, 2)$.

A negative scale factor puts the image on the opposite side of the centre, and $\frac{1}{2}$ makes it half the size. Take each corner's position relative to $(-1, 2)$, multiply by $-\frac{1}{2}$, and add $(-1, 2)$ back. For the corner $(1, 5)$:

$$(1, 5) - (-1, 2) = (2, 3) \quad \rightarrow \quad -\tfrac{1}{2}(2, 3) = (-1, -1.5) \quad \rightarrow \quad (-2, 0.5)$$

The other corners $(1, 1)$ and $(3, 1)$ become $(-2, 2.5)$ and $(-3, 2.5)$.

Vectors

We have already learned how vectors are used to perform translations on shapes and points. Now, let's learn a little more about them.

Addition and subtraction

You can add and subtract vectors by adding or subtracting the top numbers and the bottom numbers:

$$\begin{pmatrix}7\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}9\\3\end{pmatrix} \qquad \begin{pmatrix}7\\2\end{pmatrix} - \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}$$

Multiplication with a scalar

You can also multiply an integer (a scalar) with a vector. Multiply both numbers:

$$-3\begin{pmatrix}5\\-3\end{pmatrix} = \begin{pmatrix}-15\\9\end{pmatrix}$$

Magnitude of a vector

Magnitude means "how long" the vector is: its length or size. The formula for finding the magnitude of a vector is:

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

TRY IT

Find the magnitude of the vector $\begin{pmatrix}3\\4\end{pmatrix}$.

$$|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5$$

Note Academy

NOTE

The magnitude of a vector is represented by the modulus of that vector. For example, the magnitude of $\mathbf{a}$ is $|\mathbf{a}|$.

Vector geometry

Vectors can be represented using arrows. For example, the vector $\overrightarrow{OA} = \begin{pmatrix}2\\3\end{pmatrix}$ (from the origin to the point A) can be represented on the graph as:

If two vectors are scalar multiples of each other, they are **parallel**. For example, if $\overrightarrow{OA} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}1\\1.5\end{pmatrix}$, then they are parallel (because $\overrightarrow{OA} = 2\overrightarrow{BC}$).

Three points A, B and C are **collinear** (fall on the same line) if the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. Because the two vectors share the point A, being parallel is enough to put all three points on one straight line.

Here $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = 2\mathbf{a}$. They point in the same direction, so A, B and C are collinear.

Some questions may ask you to calculate vectors based on ratios.

TRY IT

$\overrightarrow{OB}$ is a vector such that $\overrightarrow{OB} : \overrightarrow{OC} = 1 : 3$. Find $\overrightarrow{OB}$. $\overrightarrow{OC}$ has already been drawn. Draw $\overrightarrow{OB}$ as well.

We can deduce from the diagram that $\overrightarrow{OC} = \begin{pmatrix}6\\3\end{pmatrix}$. The ratio says that $\overrightarrow{OB}$ is one part for every 3 parts of $\overrightarrow{OC}$, so we divide $\overrightarrow{OC}$ by 3 to get $\overrightarrow{OB}$:

$$\overrightarrow{OB} = \tfrac{1}{3}\overrightarrow{OC} = \tfrac{1}{3}\begin{pmatrix}6\\3\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

Note Academy