Number

Types of numbers, HCF and LCM, sets and Venn diagrams, powers and roots, fractions, decimals and percentages, indices, standard form, estimation and bounds, ratio, rates, interest, surds and rationalising the denominator.

Types of numbers

Natural numbers

- All naturally occurring numbers: the positive whole numbers.
- 0 is not included.
- $1, 2, 3, 4, 5, 6, \ldots$

Integers

- All positive and negative whole numbers.
- 0 is included.
- $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

Prime numbers

- Numbers that can only be evenly divided by themselves and 1.
- In short, a prime number only has 2 factors: itself and 1.
- 1 is not a prime number.
- $2, 3, 5, 7, 11, \ldots$

Square numbers

- A square number is a number you get when you multiply a number by itself.
- For example, 4 (because 2×2), 9 (because 3×3), 16, 25, 36, and so on.

Cube numbers

- A cube number is a number you get when you multiply a number by itself 2 times (three of the number are multiplied).

Note Academy

- For example, 8 (because $2 \times 2 \times 2$), 27 (because $3 \times 3 \times 3$), and so on.

Rational and irrational numbers

- A rational number is any number whose decimal stops or repeats.
- It is a number that can be written as a fraction.
- For example, 2, 3.1, $\frac{1}{3}$ $(0.33333\ldots)$, $-\frac{3}{5}$.
- An irrational number cannot be written as a fraction.
- Its decimal goes on forever and never repeats.
- For example, π $(3.14159265\ldots)$, $\sqrt{2}$.

> **Easy way to remember: rational means "ratio", so it can be written as a neat fraction. Irrational means "irritating decimal", so it goes on forever with no pattern.**

Highest common factor (HCF)

How to find the HCF:

1. Prime factorise each number.
2. Choose the **lowest** power of each common factor.
3. Multiply them.

> **TRY IT**
>
> Find the HCF of 24 and 36.

$24 = 2^3 \times 3^1$ and $36 = 2^2 \times 3^2$.

Smallest powers: 2^2 and 3^1. Hence HCF $= 2^2 \times 3 = 12$.

> **TRY IT**
>
> Find the HCF of 60, 72 and 108.

Divide all three by a number that goes into all of them, until none does:

Note Academy

Division	60	72	108
$\div 2$	30	36	54
$\div 2$	15	18	27
$\div 3$	5	6	9

HCF $= 2 \times 2 \times 3 = 12$.

Lowest common multiple (LCM)

How to find the LCM:

1. List the prime factors of each number.
2. Choose the **highest** power of all prime factors (they don't need to be common factors).
3. Multiply them together.

> **TRY IT**
>
> Find the LCM of 24 and 36.

- $24 = 2^3 \times 3^1$
- $36 = 2^2 \times 3^2$
- Highest powers: 2^3 and 3^2
- Hence LCM $= 2^3 \times 3^2 = 72$

Reciprocals

- A reciprocal is what you get when you flip a number upside down, especially a fraction.
- For example, $4 \longrightarrow \frac{1}{4}$ and $\frac{3}{5} \longrightarrow \frac{5}{3}$.

Practice questions

> **TRY IT**
>
> Q1. Tick all the rational numbers: -2, 0.75, $\sqrt{2}$, π, $\frac{1}{3}$, $0.666\ldots$

Rational: -2, 0.75, $\frac{1}{3}$ and $0.666\ldots$. Not rational: $\sqrt{2}$ and π.

Note Academy

TRY IT

Q2. Express 96 as a product of its prime factors.

Start dividing 96 by the smallest prime number and keep dividing by prime numbers until it can no longer be divided further:

- $96 \div 2 = 48$
- $48 \div 2 = 24$
- $24 \div 2 = 12$
- $12 \div 2 = 6$
- $6 \div 2 = 3$

So $96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 = 2^5 \times 3$.

A harder HCF and LCM question

TRY IT

Written as the product of its prime factors, $360 = 2^3 \times 3^2 \times 5$. (a) Write 108 as the product of its prime factors. (b) Find the lowest common multiple of 108 and 360. Give your answer as the product of its prime factors. (c) Find the smallest positive integer k such that $360k$ is a cube number.

(a) Divide by the smallest prime numbers: $108 \div 2 = 54$, $54 \div 2 = 27$, $27 \div 3 = 9$, $9 \div 3 = 3$, $3 \div 3 = 1$. So $108 = 2^2 \times 3^3$.

(b) The LCM takes the highest powers of all primes involved.

- $108 = 2^2 \times 3^3$
- $360 = 2^3 \times 3^2 \times 5$
- Highest power of 2: 2^3. Highest power of 3: 3^3. 5 appears only in 360, so take 5.

Hence LCM $= 2^3 \times 3^3 \times 5$.

(c) We need k so that $360 \times k$ is a perfect cube. For a cube number, all prime powers must be multiples of 3.

- 2^3: already a multiple of 3.
- 3^2: needs 1 more 3 to become 3^3.
- 5: needs 2 more 5s to become 5^3.

So we must multiply 360 by $3 \times 5^2 = 3 \times 25 = 75$. The smallest value of k is **75**.

Note Academy

TRY IT

The numbers 225 and 540, written as products of their prime factors, are $225 = 3^2 \times 5^2$ and $540 = 2^2 \times 3^3 \times 5$. Find the smallest positive integer value of n for which $225n$ is a multiple of 540.

To find the smallest n so that $225 \times n$ is a multiple of 540, $225n$ must have all the prime factors of 540, with the same or bigger powers of each prime.

- 225 has no 2s at all, but 540 has two 2s (2^2), so we need to multiply by $2^2 = 4$.
- 225 has 3^2, but 540 has 3^3, so we are missing one more 3. Multiply by 3.
- 225 already has enough 5s (it has 5^2 and 540 only has one 5).

Multiply all the missing pieces together: $4 \times 3 = 12$. The smallest value of n is **12**.

Sets

A set is a collection of items (called elements or members) that are grouped together because they have something in common.

- We usually write sets using curly brackets $\{\ \}$, and separate each element with a comma.
- Example: set $A = \{2, 4, 6, 8\}$ is a set of even numbers.

Symbol	Meaning	Example
$n(A)$	Number of elements in set A	$n(\{2, 4, 6\}) = 3$
$\in$	"is an element of"	$3 \in \{1, 2, 3, 4\}$
$\notin$	"is not an element of"	5 $\notin$ {1, 2, 3}
A'	Complement of set A (everything not in A)	If $A = \{1, 2\}$, A' is everything else in the universal set
$\emptyset$	The empty set (no elements)	$A = \emptyset$ means A has no members
Universal set	The set of all possible elements in that situation	Shown as a big rectangle in a Venn diagram
$A \subseteq B$	A is a subset of B (all elements in A are also in B)	$A = \{1, 2\}$, $B = \{1, 2, 3\}$, so $A \subseteq B$
A $\not\subseteq$ B	A is not a subset of B	A = {1, 4}, B = {1, 2, 3}, so A $\not\subseteq$ B
$A \cup B$	Union: all elements in A or B (or both)	$A = \{1, 2\}$, $B = \{2, 3\}$, so $A \cup B = \{1, 2, 3\}$
$A \cap B$	Intersection: elements in both A and B	$A = \{1, 2\}$, $B = \{2, 3\}$, so $A \cap B = \{2\}$

Note Academy

Set builder notation

Instead of listing every element, we can describe the rule: $\{x : \text{condition about } x\}$ means "the set of all x such that x follows a certain rule".

- $A = \{x : x \text{ is a natural number}\}$: A is the set of all natural numbers.
- $B = \{(x, y) : y = mx + c\}$: B is the set of points that lie on the line $y = mx + c$.
- $C = \{x : a \leq x \leq b\}$: C is the set of numbers between a and b, including a and b.
- $D = \{a, b, c, \dots\}$: D is a set containing the elements $a, b, c, \dots$

Venn diagrams

- Venn diagrams are pictures that show how sets relate to each other.
- They use circles inside a rectangle (which represents the universal set).

Powers and roots

Squares and cubes have been introduced before.

The **square root** is the opposite of squaring. It means: what number multiplied by itself gives this value?

Cube root means: what number multiplied by itself 3 times gives this? For example, $\sqrt[3]{64} = 4$ because $64 = 4 \times 4 \times 4$.

You can also work with:

- Higher powers like $2^4 = 2 \times 2 \times 2 \times 2 = 16$.
- Higher roots like $\sqrt[4]{81} = 3$ because $3 \times 3 \times 3 \times 3 = 81$.

Note Academy

Number	Square
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225

Number	Cube
1	1
2	8
3	27
4	64
5	125
10	1000

NOTE

Unfortunately, you have to remember all these.

Note Academy

Fractions, decimals and percentages

Type	What it means	Example
Proper	Top number is smaller than bottom (less than 1)	$\frac{3}{5}$
Improper	Top number is bigger than or equal to bottom	$\frac{7}{4}$
Mixed number	A whole number with a fraction	$1\frac{2}{4}$

A decimal is another way of writing part of a whole number using a decimal point. For example, $\frac{1}{2} = 0.5$ and $\frac{3}{4} = 0.75$.

Recurring decimals

Some decimals repeat forever. The notation puts a dot or bar over the repeating digits.

- $0.1\dot{7} = 0.17777\ldots$
- $0.\overline{123} = 0.123123123\ldots$
- $0.1\dot{2}\dot{3} = 0.1232323\ldots$

Converting percentages

Percent means "per 100". Hence 40% means 40 for every 100. The percentage symbol just means divided by 100.

Conversion	Example
Fraction to decimal (make the denominator 100)	$\frac{1}{4} = \frac{25}{100} = 0.25$
Decimal to fraction	$0.75 = \frac{75}{100} = \frac{3}{4}$
Fraction to %	$\frac{2}{5} = \frac{40}{100} = 0.4 = 40\%$
% to fraction	$20\% = \frac{20}{100} = \frac{1}{5}$
Decimal to % (just multiply by 100)	$0.6 \rightarrow 0.6 \times 100 = 60\%$
% to decimal	$30\% = \frac{30}{100} = 0.3$

Note Academy

Ordering and BODMAS

Symbol	Meaning	Example
$=$	Equal to	$5 = 5$
$\neq$	Not equal to	$3 \neq 7$
$>$	Greater than	$9 > 4$
$<$	Less than	$2 < 6$
$\geq$	Greater than or equal to	$7 \geq 7$
$\leq$	Less than or equal to	$5 \leq 9$

NOTE

If you are confused between the greater than and less than symbols, remember that the sign is the mouth of an alligator, and it always eats the bigger number.

BODMAS

- **B**rackets
- **O**rders (powers and roots)
- **D**ivision
- **M**ultiplication
- **A**ddition
- **S**ubtraction

Follow this order while simplifying long expressions (left to right).

TRY IT

Work out $(6+4) \div 2^2 + 3 \times (5-2)$.

$(6+4) \div 2^2 + 3 \times (5-2) \rightarrow 10 \div 2^2 + 3 \times 3 \rightarrow 10 \div 4 + 3 \times 3 \rightarrow 2.5 + 9 = 11.5$

Adding and subtracting fractions

First, let's learn how to convert the different types of fractions into other forms.

Mixed to improper. Example: $2\frac{1}{3}$.

Note Academy

1. Multiply the whole number by the denominator: $2 \times 3 = 6$.
2. Add the numerator: $6 + 1 = 7$.
3. Keep the same denominator. Answer $= \frac{7}{3}$.

Improper to mixed. Example: $\frac{9}{4}$.

1. Divide the top by the bottom: $9 \div 4 = 2$ with remainder 1.
2. Write the remainder as the numerator. Answer: $2\frac{1}{4}$.

> TRY IT
>
> Add $\frac{9}{4}$ and $\frac{3}{5}$.

Make the denominators the same by multiplying each fraction by the other's denominator:

$$\frac{9\times5}{4\times5} + \frac{3\times4}{5\times4} = \frac{45}{20} + \frac{12}{20} = \frac{57}{20}$$

Use the same technique for subtraction as well.

> NOTE
>
> To add or subtract mixed numbers, convert them into improper fractions first.

Indices

An index (plural: indices) tells you how many times to multiply a number by itself. Example: $5^3 = 5 \times 5 \times 5 = 125$. Here, 5 is the **base**, and 3 is the **index** or power.

Types of indices you should know

- **Positive indices** mean regular multiplying: $2^4 = 2 \times 2 \times 2 \times 2 = 16$.
- **Zero index:** any number to the power of 0 is always 1: $7^0 = 1$.
- **Negative indices:** flip the number (take the reciprocal): $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

Rule	What it means	Example
Multiply: add the powers	$a^m \times a^n = a^{m+n}$	$2^2 \times 2^3 = 2^5$
Divide: subtract the powers	$a^m \div a^n = a^{m-n}$	$3^3 \div 3 = 3^2$
Power of a power: multiply the powers	$(a^m)^n = a^{mn}$	$(2^3)^2 = 2^6$
Fractional power: root the number	$a^{1/n} = \sqrt[n]{a}$	$8^{1/3} = \sqrt[3]{8}$

Note Academy

TRY IT

Simplify $27^{-1/3}$.

$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}$

TRY IT

Solve $2^{-2} \times 2^{5}$.

Use the same base rule: add the powers if multiplied, subtract if divided. Since we are multiplying, we add the powers:

$2^{-2+5} = 2^{3} = 8$

TRY IT

Simplify $\left(\frac{18x^3y}{6xy^3}\right)^{-2}$.

Work inside the brackets first (BODMAS), using $\frac{x^3}{x} = x^{3-1}$ and $\frac{y}{y^3} = y^{1-3}$:

$(3x^2y^{-2})^{-2} = 3^{-2}\,x^{2\times(-2)}\,y^{(-2)\times(-2)} = \frac{1}{3^2}\,x^{-4}\,y^{4} = \frac{y^4}{9x^4}$

TRY IT

Find b if $16^{b} = 4$.

Write 16 as a power of 4: $(4^2)^b = 4$, so $4^{2b} = 4^1$. For the same base, we can take the powers as a separate equation: $2b = 1$, so $b = \frac{1}{2}$.

Standard form

Standard form is also called scientific notation. It helps us write very big or very small numbers in a simple way, using powers of 10. It looks like this:

$a \times 10^{n}$

where a is a number between 1 and 10 ($1 \leq a < 10$) and n is an integer (it can be positive or negative).

How to convert a big number to standard form

1. Move the decimal point so there is only one digit before it.
2. Count how many places you moved it. That is the power of 10.

Note Academy

Example: $53000 \to 5.3 \times 10^4$ (the decimal point moved four places to the left).

How to convert a small number to standard form

1. Move the decimal point after the first non-zero digit.
2. Count the places moved. This time the power of 10 is negative.

Example: $0.0076 \to 7.6 \times 10^{-3}$ (the decimal point moved three places to the right).

Estimation

Estimation helps us get a rough idea of an answer without needing to work it out exactly. It is useful when we want to save time and check if an answer is sensible, or when an exact value is not needed.

You may be asked to round:

- to the nearest 10, 100, 1000 (place value)
- to a certain number of decimal places
- to a certain number of significant figures

Significant figures are digits that carry meaning.

Round 5764 to:

- the nearest thousand: 6000
- 1 significant figure: 6000
- 2 significant figures: 5800

Round 0.004678 to:

- 1 significant figure: 0.005
- 2 significant figures: 0.0047

Note Academy

> **TRY IT**
>
> Estimate $\dfrac{41.3}{9.79 \times 0.765}$.

Step 1. Round each number to 1 significant figure: $41.3 \approx 40$, $9.79 \approx 10$, $0.765 \approx 0.8$.

Step 2. Estimate: $\dfrac{40}{10 \times 0.8} = \dfrac{40}{8} = 5$.

Limits of accuracy

- When we round a number, the real value could be slightly more or less than the rounded number.
- So we talk about the smallest and largest the actual value could be.

- These are called the **lower bound** and the **upper bound**.

Example. The number is 72, rounded to the nearest whole number. Nearest whole number means we round to the nearest 1. Lower bound $= 71.5$ and upper bound $= 72.5$.

When numbers are used in calculations, we can calculate the maximum and minimum possible answers using the bounds.

> TRY IT
>
> Find the upper bound of the area. Length $= 8.4$ cm (rounded to 1 d.p.) and width $= 5.1$ cm (rounded to 1 d.p.).

- Length upper bound $= 8.45$
- Width upper bound $= 5.15$
- Maximum area $= 8.45 \times 5.15 = 43.5175\ \text{cm}^2$

Ratio and proportion

A ratio compares two or more values. It shows how much of one thing there is compared to another.

Simplifying ratios

Example: $20 : 30 : 40$. Find the HCF (highest common factor), which is 10, and divide each part by 10 to get $2 : 3 : 4$.

> NOTE
>
> Always divide all parts of the ratio by the same number.

Dividing a quantity in a ratio

Example: divide 90 in the ratio $2 : 3$.

- Step 1: add the parts: $2 + 3 = 5$.
- Step 2: divide 90 by 5: $90 \div 5 = 18$.
- Step 3: multiply: first part $2 \times 18 = 36$ and second part $3 \times 18 = 54$.

Answer: 36 and 54.

Note Academy

Using ratios in real life

> **TRY IT**
>
> A recipe uses flour and sugar in the ratio $3:2$. You want to use 9 cups of flour. How much sugar?

Flour : sugar is $3:2 = 9:x$. Cross-multiplying, $3x = 18$, so $x = 6$ cups of sugar.

> **TRY IT**
>
> On a map with a scale of $1:1000$, how long in real life is 5 cm on the map?

That means 1 cm on the map is 1000 cm in real life. So 5 cm on the map is $5 \times 1000 = 5000$ cm.

Rates

A rate tells us how much something happens over a certain time or space. We usually see it written like km/h (kilometres per hour) or g/cm³ (grams per cubic centimetre).

> **TRY IT**
>
> Maria earns $72 for 6 hours of work. What is her hourly rate?

Hourly rate $= \frac{72}{6} = 12$ dollars per hour.

> **TRY IT**
>
> A block has mass 200 g and volume 100 cm³. Calculate its density.

Density $= \frac{\text{mass}}{\text{volume}} = \frac{200}{100} = 2\ \text{g/cm}^3$.

> **TRY IT**
>
> A cyclist travels 45 km in 3 hours 45 minutes. Calculate his average speed.

3 hours 45 minutes $= 3.75$ hours. Speed $= \frac{\text{distance}}{\text{time}} = \frac{45}{3.75} = 12$ km/h.

For questions about speed, distance and time, remember this triangle, with distance (D) on top and speed (S) and time (T) underneath:

- To find **distance**, multiply speed and time: $D = S \times T$.
- To find **speed**, divide distance by time: $S = \frac{D}{T}$.
- To find **time**, divide distance by speed: $T = \frac{D}{S}$.

Note Academy

Percentages

Calculate a given percentage. Find 30% of 80: $\frac{30}{100} \times 80 = 24$.

Express one quantity as a percentage of another. What percentage is 20 out of 50? $\frac{20}{50} \times 100 = 40\%$.

Percentage increase and decrease

> **TRY IT**
>
> A jacket costs \$50, and its price increases by 10%. What is its final price?

Increase $= 50 \times \frac{10}{100} = 5$ dollars. Final price $= 50 + 5 = 55$ dollars.

> **TRY IT**
>
> A phone costs \$200, and gets a discount of 20%. What is the final price?

Discount $= \frac{20}{100} \times 200 = 40$ dollars. Final price $= 200 - 40 = 160$ dollars.

Simple and compound interest

Simple interest

$$\text{Interest} = \frac{P \times R \times T}{100}$$

where P is the principal (starting money), R is the rate (%) and T is the time (years).

> **TRY IT**
>
> \$1000 was left in the bank for 2 years at an interest rate of 5%. How much money is available at the end of the 2 years?

Interest $= \frac{1000 \times 5 \times 2}{100} = 100$ dollars. Total money available $=$ interest $+$ principal $= 100 + 1000 = 1100$ dollars.

Compound interest

$$F = P\left(1 + \frac{r}{100}\right)^n$$

where F is the final money available, P is the principal, r is the rate and n is the number of terms (if the compounding is yearly, n is the number of years).

Note Academy

TRY IT

Someone invests $100 in a bank at 10% compound interest for 2 years. How much money will they have after 2 years?

$$F = 100\,(1 + \tfrac{10}{100})^2 = 121 \text{ dollars}$$

The principal for simple interest stays the same each year, whereas the principal for compound interest changes every year.

In the simple interest question above, there is a total interest of $100 over 2 years, so the same amount ($50) is added each year.

- Compound interest means you earn interest not just on your original money, but also on the interest you already earned.
- So in this question, someone puts $100 in the bank, and the bank gives 10% interest every year.
- After the first year, 10% of $100 is $10, so the total becomes $110. In the second year, the 10% is now calculated on $110, which gives $11 more. That makes the total after two years $121.

So the amount keeps growing faster, because each year the interest is added to the new total, not just the starting amount.

NOTE

For questions about interest, make sure you identify the type so that you know which formula to use. Usually, questions about simple interest will have the words "simple interest" in them. The others will be about compound interest.

Note Academy

Reverse percentages

TRY IT

The selling price is $120 after a 20% increase. What was the cost price?

In the previous questions, we calculated the change and then added or subtracted it from the initial cost. Using this idea, we get:

$$\text{Initial cost} \times \frac{100+\text{percentage change}}{100} = \text{Final price}$$

So:

$$\text{Cost price} = \frac{120}{(20+100)\%} = \frac{120}{1.2} = 100 \text{ dollars}$$

If the question said that the final price was after a 20% **decrease**, the percentage change would be negative, which gives:

$$\text{Cost price} = \frac{120}{(-20+100)\%} = \frac{120}{0.8} = 150 \text{ dollars}$$

Money

> **TRY IT**
>
> If €1 = Rs. 310, how much is €7 in rupees?

Use the ratio method. Euro : PKR is $1 : 310 = 7 : x$, so $x = 7 \times 310 = \text{Rs. } 2170$.

Surds

> **NOTE**
>
> **Extended only.** This topic is in the Extended syllabus, so it is asked in Papers 2 and 4. If you sit the Core papers you do not need it.

A surd is a square root (or other root) that can't be simplified into a whole number. It just keeps going and going, like a never-ending decimal. Think of $\sqrt{2}$ or $\sqrt{3}$: you can't write them as exact decimals.

Simplifying surds

We simplify surds by finding square numbers inside them.

- $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$
- $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$

We can only add surds that have the same root.

- $\sqrt{3} + \sqrt{5}$: not possible to combine (the roots are different, so the answer stays as $\sqrt{3} + \sqrt{5}$).
- $\sqrt{5} + \sqrt{5} = 2\sqrt{5}$
- $\sqrt{8} + \sqrt{2} = \sqrt{2 \times 2 \times 2} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$

Rationalising the denominator

> **TRY IT**

Note Academy

Rationalise $\frac{1}{1+\sqrt{3}}$.

Multiply top and bottom by the conjugate, which is $(1-\sqrt{3})$:

$$\frac{1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1-\sqrt{3}}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{1-\sqrt{3}}{1-3} = \frac{1-\sqrt{3}}{-2} = \frac{\sqrt{3}-1}{2}$$

NOTE

The conjugate is the same expression as the denominator but with a changed sign.

TRY IT

Rationalise $\frac{10}{\sqrt{5}}$.

$$\frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$$

For normal denominators, multiply the surd in the denominator with the numerator and the denominator.

Note Academy