Algebra and Graphs

Algebraic manipulation and factorising, algebraic fractions, equations and inequalities, sequences, proportion, functions, and drawing and reading graphs.

Introduction to algebra

Algebra is like a maths game where we use letters to stand for numbers we don't know yet.

For example, if I say "I have x candies, and I eat 3", then $x - 3$ means how many candies are left. It is just a way to represent patterns or unknown values.

> **TRY IT**
>
> If $x = 4$, what is the value of $2x + 3$?

Replace x with 4:

$$2 \times 4 + 3 = 8 + 3 = 11$$

Algebraic manipulation

Simplify expressions by collecting like terms

Like terms are terms that have the same letters and powers. You can add or subtract them.

> **TRY IT**
>
> Simplify $2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$.

Group like terms:

$$(2a^2 + 5a^2) + (3ab - 9ab) + (-1 + 4) = 7a^2 - 6ab + 3$$

Expand products of algebraic expressions

To expand means to get rid of brackets.

> **TRY IT**
>
> Expand $3x(2x - 4y)$.

Note Academy

$3x \times 2x - 3x \times 4y = 6x^2 - 12xy$

> **TRY IT**
>
> Expand $(x-2)(x+3)(2x+1)$.

First do two brackets with the FOIL method (First, Outside, Inside, Last) so that you do not forget any term, for $(x-2)(x+3)$:

- First: $x \times x = x^2$
- Outside: $x \times 3 = +3x$
- Inside: $-2 \times x = -2x$
- Last: $-2 \times 3 = -6$

Now add it all up: $x^2 + 3x - 2x - 6 = x^2 + x - 6$.

So now the question becomes $(x^2 + x - 6)(2x + 1)$. Multiply every term in the first bracket by every term in the second:

$(x^2 \times 2x) + (x \times 2x) + (-6 \times 2x) + (x^2 \times 1) + (x \times 1) + (-6 \times 1)$

$= 2x^3 + 2x^2 - 12x + x^2 + x - 6 = 2x^3 + 3x^2 - 11x - 6$

Factorise by taking out common factors

Factorising is the opposite of expanding.

> **TRY IT**
>
> Factorise $9x^2 + 15xy$.

What do both terms have in common? They both have x, and they are both divisible by 3, so you take that out:

$9x^2 + 15xy = 3x(3x + 5y)$

Factorise special expressions

Type 1: $ax^2 + bx + kay + kby$ (grouping)

> **TRY IT**
>
> Factorise $2x^2 + 4xy + 3x + 6y$.

Note Academy

Group it like this: $(2x^2 + 4xy) + (3x + 6y)$. Now take out common factors:

$2x(x + 2y) + 3(x + 2y)$

Since $(x + 2y)$ is in both, the final answer is $(2x + 3)(x + 2y)$.

Type 2: $a^2x^2 - b^2y^2$ (difference of squares)

> $a^2 - b^2 = (a + b)(a - b)$

> **TRY IT**
>
> Factorise $16x^2 - 25y^2$.

$16x^2 - 25y^2 = (4x)^2 - (5y)^2 = (4x + 5y)(4x - 5y)$

Type 3: $a^2 + 2ab + b^2$ (perfect square)

> $a^2 + 2ab + b^2 = (a + b)^2$ **and** $a^2 - 2ab + b^2 = (a - b)^2$

For example, $x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2$, and in the same way $x^2 - 6x + 9 = (x - 3)^2$.

Type 4: completing the square

> **TRY IT**
>
> Write $x^2 + 6x + 5$ in the form $(x + a)^2 + b$.

Focus on $x^2 + 6x$.

- Take half of 6, which is 3. This is because $(x + a)^2 = x^2 + 2ax + a^2$.
- So $(x + 3)^2 = x^2 + 6x + 9 = x^2 + 2(3)x + 9$.

> **Basically, just divide the coefficient of x by 2, and add and subtract its square.**

By adding and subtracting 9, the expression remains the same:

$x^2 + 6x + 5 + 9 - 9 = (x^2 + 6x + 9) - 9 + 5 = (x + 3)^2 - 4$

A more complex question: $2x^2 + 8x + 3$. Factor out 2 from the first two terms (to make the coefficient of x^2 one), then complete the square inside the bracket:

$2(x^2 + 4x) + 3 = 2\left[x^2 + 4x + 4 - 4\right] + 3 = 2\left[(x + 2)^2 - 4\right] + 3$

Note Academy

$= 2(x+2)^2 - 8 + 3 = 2(x+2)^2 - 5$

Type 5: mid-term breaking

> **TRY IT**
>
> Factorise $2x^2 + 11x + 5$.

1. Find two numbers that multiply to $2 \times 5 = 10$ and add up to 11. These are 1 and 10.
2. Break the middle: $2x^2 + 10x + x + 5$.
3. Group: $(2x^2 + 10x) + (x + 5)$.
4. Factor: $2x(x+5) + 1(x+5)$.
5. Final factors: $(2x+1)(x+5)$.

Algebraic fractions

Addition and subtraction

Make the denominators the same.

> **TRY IT**
>
> Simplify $\frac{x}{3} + \frac{x-4}{2}$.

$\frac{2x}{6} + \frac{3(x-4)}{6} = \frac{2x+3x-12}{6} = \frac{5x-12}{6}$

Multiplication

Just multiply the numerators and denominators.

$\frac{2x}{3} \times \frac{x-4}{2} = \frac{2x(x-4)}{6}$

Division

Flip the second fraction and multiply.

$\frac{2x}{3} \div \frac{x-4}{2} = \frac{2x}{3} \times \frac{2}{x-4} = \frac{4x}{3(x-4)}$

Note Academy

Factorise and simplify rational expressions

TRY IT

Simplify $\dfrac{x^2-2x}{x^2-5x+6}$.

Factorise the top: $x^2-2x=x(x-2)$.

Factorise the bottom with mid-term breaking: we need two numbers that multiply to 6 and add to -5. They are -3 and -2.

$$x^2-3x-2x+6=x(x-3)-2(x-3)=(x-3)(x-2)$$

Now cancel the common factor $(x-2)$:

$$\frac{x(x-2)}{(x-3)(x-2)}=\frac{x}{x-3}$$

TRY IT

Simplify $\dfrac{1}{x-2}+\dfrac{x+1}{x-3}$.

Make the denominators the same:

$$\frac{1\times(x-3)}{(x-2)(x-3)}+\frac{(x+1)(x-2)}{(x-3)(x-2)}=\frac{(x-3)+(x+1)(x-2)}{(x-2)(x-3)}$$

Leave the answer like this, because it is simplified.

Indices (again)

TRY IT

Simplify $\dfrac{2}{5x^2}\div 2x^{-3}$.

A negative index means a reciprocal, so $2x^{-3}=\dfrac{2}{x^3}$. Dividing by a fraction means multiplying by its flip:

$$\frac{2}{5x^2}\div\frac{2}{x^3}=\frac{2}{5x^2}\times\frac{x^3}{2}=\frac{x}{5}$$

Equations

An equation is used to describe the relationship between two or more things. For example, the area of a rectangle can be written like this: Area = Length × Width. If the length is 5 cm and the width is 6 cm, the area is $5\times 6=30\ \text{cm}^2$.

Note Academy

Solving linear equations

We can modify equations without changing the actual relationships.

> **TRY IT**
>
> Solve $3x + 4 = 10$.

Subtract 4 from both sides: $3x = 10 - 4 = 6$. Divide both sides by 3: $x = 2$.

> **To check your answer, plug it back into the equation: $3(2) + 4 = 10$ ✓. Since both sides are equal, the answer is correct.**

A harder question:

> **TRY IT**
>
> Solve $2(x - 3) = 3(x + 2)$.

1. Expand the brackets: $2x - 6 = 3x + 6$.
2. Take the 6 on the right to the left (-6 on the left): $2x - 6 - 6 = 3x$, so $2x - 12 = 3x$.
3. Take the $2x$ to the right ($-2x$ on the right): $-12 = 3x - 2x$.

So $x = -12$.

What you CAN do	What you CANNOT do
Add the same number to both sides	Add a number to only one side
Subtract the same number from both sides	Subtract from only one side without balancing
Multiply both sides by the same number (except 0)	Multiply only one side without doing it to the other
Divide both sides by the same non-zero number	Divide only one side
Expand brackets on both sides	Expand only one side incorrectly
Collect like terms carefully	Mix up unlike terms (e.g. add x and x^2)
Swap sides if you want (keep the equals!)	Flip parts without keeping the balance
Solve step by step, keeping it neat	Skip steps and get confused
Check your answer by substituting back	Assume you're always right without checking

Note Academy

The very important table (memorise!)

Operation on one side	On the other side
Add a number (+)	Subtract the same number (−)
Subtract a number (−)	Add the same number (+)
Multiply by a number (×)	Divide by the same number (÷)
Divide by a number (÷)	Multiply by the same number (×)
Add a variable term ($+x$)	Subtract the same variable term ($-x$)
Subtract a variable term ($-x$)	Add the same variable term ($+x$)
Multiply by a negative number	Divide by the same negative number
Take a square root ($\sqrt{}$)	Square both sides (^{2})
Square a variable (^{2})	Take the square root ($\sqrt{}$)

Solving quadratic equations

Previously we learnt different techniques to factorise expressions so that we can solve different types of equations. The quadratic formula solves equations that cannot be factorised or solved through mid-term breaking.

Always compare with $ax^2 + bx + c = 0$. For example, $x^2 + 3x - 10 = 0$ gives $a = 1$, $b = 3$, $c = -10$.

The quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug all the values into this equation. You will always get two answers (sometimes they might be the same).

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-10)}}{2(1)} = \frac{-3 \pm \sqrt{49}}{2}$$

$$x = \frac{-3+7}{2} = 2 \qquad \text{or} \qquad x = \frac{-3-7}{2} = -5$$

Now let's solve an equation using the completing the square method.

TRY IT

Solve $x^2 + 6x + 5 = 0$.

Note Academy

$x^2 + 6x = -5$

$x^2 + 6x + 9 = -5 + 9$

$(x + 3)^2 = 4$

$x + 3 = \pm 2$

So $x + 3 = 2$ gives $x = -1$, and $x + 3 = -2$ gives $x = -5$.

Solving fractional equations

> **TRY IT**
>
> Solve $\frac{2}{x-1} + \frac{5}{x-2} = 2$.

Make the denominators the same:

$\frac{2(x-2)+5(x-1)}{(x-1)(x-2)} = 2$

The denominator is being divided over the right side, so it will be multiplied there:

$2(x-2) + 5(x-1) = 2(x-1)(x-2)$

Expand both sides:

$2x - 4 + 5x - 5 = 2(x^2 - 3x + 2) = 2x^2 - 6x + 4$

Take everything to one side and group the same terms:

$2x^2 - 13x + 13 = 0$

Now use the quadratic formula with $a = 2$, $b = -13$, $c = 13$:

$x = \frac{13 \pm \sqrt{169-104}}{4} = \frac{13 \pm \sqrt{65}}{4}$

So $x = 5.27$ or $x = 1.23$ (to 3 significant figures).

Solving simultaneous equations

Now we will find the values of two unknowns using two different equations:

$2x + 3y = 12 \qquad 4x - y = 5$

Elimination method. Our goal is to cancel one of the variables by adding or subtracting the equations. Multiply the second equation by 3 to make the coefficients of y match: $(4x - y = 5) \times 3 \Rightarrow 12x - 3y = 15$.

Note Academy

Now add the two equations:

$(2x + 3y) + (12x - 3y) = 12 + 15 \quad \Rightarrow \quad 14x = 27 \quad \Rightarrow \quad x = \frac{27}{14}$

Substitute x into one of the original equations (it is easier to use the first one):

$2\left(\frac{27}{14}\right) + 3y = 12 \quad \Rightarrow \quad \frac{27}{7} + 3y = 12 \quad \Rightarrow \quad 3y = 12 - \frac{27}{7} = \frac{57}{7} \quad \Rightarrow \quad y = \frac{57}{21} = \frac{19}{7}$

Substitution method. Make one of the variables the subject (whichever is easier) and plug it into the other equation. From $4x - y = 5$ we get $y = 4x - 5$. Plug this into the other equation:

$2x + 3(4x - 5) = 12 \quad \Rightarrow \quad 2x + 12x - 15 = 12 \quad \Rightarrow \quad 14x = 27 \quad \Rightarrow \quad x = \frac{27}{14}$

$y = 4\left(\frac{27}{14}\right) - 5 = \frac{19}{7}$

Changing the subject of the equation

This means making another letter the main subject.

> **TRY IT**
>
> Make y the subject of $\frac{2x + 3y}{2} = z$.

$2x + 3y = 2z \quad \Rightarrow \quad 3y = 2z - 2x \quad \Rightarrow \quad 3y = 2(z - x) \quad \Rightarrow \quad y = \frac{2(z-x)}{3}$

Inequalities

Inequalities are used when the value of a variable lies in a certain range. To represent them, we use the symbols $<, >, \leq, \geq$. For example, if x can be 3 or any value above 3, we write $x \geq 3$ (x is greater than or equal to 3).

Representing inequalities on a number line

> **TRY IT**
>
> Solve and draw $-3 \leq x < 1$.

This means that x can be any value between -3 and 1, but 1 is not included.

- **Closed circle** if it is $\leq$ or $\geq$ (including the number).
- **Open circle** if it is $<$ or $>$ (strict, not including the number).

On the number line: a filled dot at -3 (because it includes -3) and an open circle at 1 (because it does NOT include 1).

Note Academy

Solving linear inequalities

You solve inequalities like you solve normal equations, except for one thing.

> **If you multiply or divide by a negative number, you flip the inequality sign.**

> TRY IT
>
> Solve $-2x + 2 < 2(x - 1)$.

$-2x + 2 < 2x - 2 \quad \Rightarrow \quad -2x - 2x < -2 - 2 \quad \Rightarrow \quad -4x < -4$

Divide both sides by -4. The sign flips because we divided by a negative number:

$x > 1$

> TRY IT
>
> Solve $-3 \leq 3x - 2 < 7$.

Solve each part separately:

- $-3 \leq 3x - 2$ gives $-3 + 2 \leq 3x$, so $-1 \leq 3x$, so $x \geq -\frac{1}{3}$.
- $3x - 2 < 7$ gives $3x < 9$, so $x < 3$.

Together: $-\dfrac{1}{3} \leq x < 3$.

Representing inequalities on graphs

In 2D graphs (with x and y), inequalities are shown as lines and shaded areas.

- Use a broken (dotted) line if it is $<$ or $>$ (not including the line).
- Use a solid line if it is $\leq$ or $\geq$ (including the line).
- Shade the unwanted region unless the question says otherwise.

> TRY IT
>
> What inequalities does the shaded region below show?

First, check for x: the region is between 1 and 3 and the line is solid. Hence $1 \leq x \leq 3$.

Similarly, y is between 2 and 4, so $2 \leq y \leq 4$.

Note Academy

Sequences

- A sequence is just a list of numbers that follow a pattern.
- Each number is called a **term**.
- For example, $2, 4, 6, 8, 10, 12, \ldots$ is a sequence where each number goes up by 2.

Type of sequence	What happens	Example	Rule or pattern
Linear	Add or subtract the same number	$3, 6, 9, 12, 15, \ldots$	Add 3 every time (common difference)
Quadratic	Second difference is constant	$1, 4, 9, 16, 25, \ldots$	Square numbers ($1^2, 2^2, 3^2, \ldots$)
Cubic	Third difference is constant	$1, 8, 27, 64, 125, \ldots$	Cube numbers ($1^3, 2^3, 3^3, \ldots$)
Exponential	Multiply or divide each time	$2, 4, 8, 16, 32, \ldots$	Multiply by 2 each time
Simple combinations	A mix of different patterns	$2, 4, 7, 11, 16, \ldots$	$+2, +3, +4, +5$ (the pattern changes each step)

Continuing sequences

> **TRY IT**
>
> What are the next three terms in $5, 8, 11, 14, _, _, _$?

The pattern is $+3$ each time, so the following numbers are $17, 20, 23$.

> **TRY IT**
>
> What are the next three terms in $1, 2, 4, 7, 11, 16, _, _, _$?

- The difference between the first and second terms is 1.
- For the next terms, it is 2, then 3, then 4, then 5.
- So the following numbers are $(16 + 6) = 22$, $(22 + 7) = 29$, $(29 + 8) = 37$.

Note Academy

Finding the nth term

The nth term is the formula to find any term in the sequence.

> **TRY IT**
>
> Find the 37th term of $3, 7, 11, 15, 19, \ldots$

First find the pattern: each time you add 4. Now derive a formula for the nth term:

$$n\text{th term} = 4n - 1$$

To find the formula for **arithmetic sequences** such as this, multiply the difference by n ($+4$) and then adjust it to match the first term (the first term is 3, and $4 \times 1 = 4$, so subtract one from each term).

The 37th term is $4 \times 37 - 1 = 147$.

- For **squared terms**, the formula will be $n\text{th term} = n^2 + c$, where c is the adjustment, like -1 in the arithmetic sequence formula.
- For **cubed terms**, it will be n^3 instead of n^2.
- For the **simple combination** above ($2, 4, 7, 11, 16, \ldots$), it will be $n\text{th term} = \frac{1}{2}n(n+1) + c$, where c is the adjustment. Here $c = 1$, because $\frac{1}{2}(1)(2) + 1 = 2$.

Proportion

Proportion means two things change in a connected way. You can write this using the $\propto$ symbol, "is proportional to".

Type	Meaning	Equation	Example
Direct proportion	As one goes up, the other goes up	$y \propto x$, so $y = kx$	More hours studied, higher marks. If $y = 3x$ then $x = 2$ gives $y = 6$
Inverse proportion	As one goes up, the other goes down	$y \propto \frac{1}{x}$, so $y = \frac{k}{x}$	More workers, less time. If $y = \frac{10}{x}$ then $x = 5$ gives $y = 2$
Square proportion	Changes with the square of x	$y \propto x^2$	y = area, x = side, so $y = x^2$
Square root	Changes with the square root of x	$y \propto \sqrt{x}$	Speed $\propto \sqrt{\text{height}}$ in physics
Cube proportion	Changes with the cube of x	$y \propto x^3$	Volume $\propto$ side^3
Cube root	Changes with the cube root of x	$y \propto \sqrt[3]{x}$	Physics or chemistry problems

Note Academy

TRY IT

y is directly proportional to x. When $x = 4$, $y = 12$. Find y when $x = 9$.

- Start with $y \propto x$, so $y = kx$.
- Use the values to find k: $12 = k \times 4$, so $k = 3$.
- Now use it to find y when $x = 9$: $y = 3 \times 9 = 27$.

TRY IT

y is inversely proportional to the square of x. When $x = 2$, $y = 5$. Find y when $x = 5$.

- Start with $y \propto \frac{1}{x^2}$, so $y = \frac{k}{x^2}$.
- Use the values to find k: $5 = \frac{k}{2^2}$, so $5 = \frac{k}{4}$, so $k = 20$.
- Now find y when $x = 5$: $y = \frac{20}{5^2} = 0.8$.

Functions

NOTE

Extended only. This topic is in the Extended syllabus, so it is asked in Papers 2 and 4. If you sit the Core papers you do not need it.

A function is like a machine: you give it a number (input), and it gives you a new number (output) based on a rule. We use special notation: $f(x)$ means "the function f of x".

If $f(x) = 3x - 5$, then $f(2) = 3(2) - 5 = 6 - 5 = 1$.

Domain and range

These are like the rules for input and output.

- **Domain:** what values you are allowed to put into the function (the inputs). Usually, it is x.
- **Range:** what values can come out (the outputs). Usually, it is $f(x)$.

TRY IT

Find the domain and range of $h(x) = 2x^2 + 3$.

Note Academy

You can plug any real number into x, so the domain is all real numbers. Because squaring never gives a negative result, the smallest output is when $x = 0$: $h(0) = 2(0)^2 + 3 = 3$. So the range is $y \geq 3$.

Some functions cannot have all real numbers as their domain. For example, $y = \sqrt{x}$ has the domain $x \geq 0$ (only zero and above), because negative numbers do not have a real square root.

Inverse functions

An inverse function reverses what the actual function did.

> **TRY IT**
>
> Find $f^{-1}(x)$ if $f(x) = 3x - 5$.

1. Replace $f(x)$ with y: $y = 3x - 5$.
2. Swap x and y: $x = 3y - 5$.
3. Make y the subject: $x + 5 = 3y$, so $y = \dfrac{x + 5}{3}$.

So $f^{-1}(x) = \dfrac{x + 5}{3}$. That means: if $f(2) = 1$, then $f^{-1}(1) = 2$.

Composite functions

This means putting one function inside another. Let $f(x) = x + 2$ and $g(x) = 3x$.

> **TRY IT**
>
> Find $g(f(x))$.

To find the composite function $g(f(x))$, take the function on the right, $f(x)$, and put it into the one on the left, $g(x)$:

$$g(f(x)) = g(x + 2) = 3(x + 2) = 3x + 6$$

> **NOTE**
>
> $gf(x)$ and $fg(x)$ are totally different things. Mind the order.

Note Academy

Graphs in practical situations

Using and interpreting graphs

Graphs are pictures that show information. Instead of using lots of words or numbers in a table, we use lines and shapes to tell a story visually.

A basic graph has:

- **x-axis** (independent variable, horizontal): usually shows time or input.
- **y-axis** (dependent variable, vertical): usually shows distance, speed, money or output.
- **Points:** little dots that show values.
- **Lines or curves:** connect the points to show how things change.

This graph shows a person's trip to the grocery store.

- In the first 20 minutes, the person travels 10 metres towards the store.
- He then takes a break for 10 minutes (because he didn't cover any distance).
- He then travels again for 10 minutes and reaches the store.
- In the last 20 minutes, the person travelled back to his home.

Plotting graphs

> **TRY IT**
>
> Plot a graph of $y = 2x - 1$ for x from -2 to 3.

First plug the values into the equation to find the values of y:

x	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5

For example, $x = -2$ gives $y = 2(-2) - 1 = -5$. Finally, plot the points and draw the graph.

Gradient

The gradient, also called the slope, is calculated using this formula:

Note Academy

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

To calculate it, first choose 2 points from the graph, for example $(1, 1)$ and $(2, 3)$. Then plug these values into the formula:

$m = \frac{3-1}{2-1} = 2$

For speed-time and distance-time graphs, remember this:

- The gradient of a **distance-time** graph = **speed** (because it shows the distance covered divided by the time taken).
- The gradient of a **speed-time** graph = **acceleration** (because it shows the change in speed divided by the time taken).

Distance-time graphs

$$\textbf{Speed} = \frac{\text{distance}}{\text{time}}$$

Using the trip graph above, we can calculate the gradient of each straight part to give the speed. Since the lines are straight, the person is travelling at constant speed.

- First 20 minutes: speed $= \frac{10}{20} = 0.5$ m/min.
- Next 10 minutes: stationary (the line is flat).
- The next 10 minutes: speed $= \frac{15 - 10}{40 - 30} = 0.5$ m/min.
- The last 20 minutes: speed $= \frac{0 - 15}{60 - 40} = -0.75$ m/min.

In the final 20 minutes, the velocity is negative because the person travels back from where he went (going in the opposite direction).

Curved lines. A curved line that gets steeper means the person is **accelerating**, because he is covering more distance in the same time interval as time goes on.

A curve that gets flatter means the person is **decelerating**: he covers less and less distance in the same time interval, so his speed is decreasing.

Speed-time graphs

Here is a speed-time graph of a car going towards a certain destination.

Note Academy

We can find the distance travelled by the car in a certain time by calculating the **area under the graph**. Divide the graph into shapes like triangles, rectangles and trapeziums.

- First 20 s (triangle): distance $= \frac{1}{2} \times 20 \times 10 = 100$ m.
- Next 10 s (rectangle): distance $= (30 - 20) \times 10 = 100$ m.
- From 30 s to 40 s (trapezium): distance $= \dfrac{(15 + 10)}{2} \times (40 - 30) = 125$ m.
- The last 20 s (triangle): distance $= \frac{1}{2} \times (60 - 40) \times 15 = 150$ m.

The total distance is $100 + 100 + 125 + 150 = 475$ m. Since the speed was positive the whole time, the car was only travelling away from its starting point.

In the first 20 s, the graph slopes upwards. This means that the car is **accelerating**, because the speed of the car is increasing. From 40 s to 60 s, the slope is downwards, which means that the car is **decelerating**.

Converting units

> **TRY IT**
>
> Convert 3000 m/s to km/h.

First write down the units in fraction form: $\dfrac{\text{m}}{\text{s}} \rightarrow \dfrac{\text{km}}{\text{h}}$.

Next, convert the units while keeping them in fraction form. There are 1000 metres in a kilometre and 3600 seconds in an hour:

$$\frac{\text{m}/1000}{\text{s}/3600} \rightarrow \times 3.6$$

If we multiply the number by 3.6, we get the final answer: $3000 \times 3.6 = 10\,800$ km/h.

Graphs of functions

Linear functions

A linear function always gives a straight line on a graph. The equation will always follow this format:

> $y = mx + c$**, where** m **is the gradient and** c **is the** y**-intercept.**

For example, $f(x) = 2x - 1$ has gradient 2, and the line crosses the y-axis at -1. It is the graph we plotted above.

Note Academy

Polynomial functions

Polynomials usually follow the format $y = ax^n$. Some examples are $y = x^2$, $y = x^3$ and $y = x^2 + 2x + 1$.

Graphs of x^2 and $-x^2$

To plot the graphs of polynomial functions, plot the points on the graph, then join them using a free hand (make a curve instead of straight lines). The graph of $y = x^2$ is symmetrical about the y-axis.

To find the turning point of the graph, convert the equation into the completed-square form.

> **TRY IT**
>
> Find the turning point of $x^2 + 6x + 3$.

$x^2 + 6x + 9 - 9 + 3 = (x + 3)^2 - 6$

The x-coordinate comes from $x + 3 = 0$, so $x = -3$. The y-coordinate is the number at the end, -6.

Turning point: $(-3, -6)$.

Graphs of cubics

Reciprocal functions

These functions have vertical and horizontal **asymptotes**. Asymptotes are lines to which the graph keeps getting closer but never touches.

- The graphs of $\frac{1}{x}$ and $-\frac{1}{x}$ are symmetrical about the line $y = x$ and $y = -x$ (the dotted line on the graph).
- They have the y-axis as the vertical asymptote and the x-axis as the horizontal asymptote.

The graphs of $\frac{1}{x^2}$ and $-\frac{1}{x^2}$ are symmetrical about the y-axis.

Exponential functions

These have the form $y = a^x$, where a is a positive number (not 1).

- The gradient of the graph keeps increasing, hence its name.
- Exponential functions always have a y-intercept of 1, except when a number is being added or subtracted.
- For example, the red line is the graph of $y = 2^x$, and the purple line is the graph of $y = 2^x - 1$ (it moves one unit down the y-axis).
- The red line has a horizontal asymptote on the x-axis ($y = 0$), and the purple one has an asymptote at $y = -1$.

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Roots, domain and range from a graph

Another important thing in graphs is the **roots** of the graph (x-intercepts). These are very helpful in finding out what type of graph it is.

> TRY IT
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> Find the domain and range for the following graph.

- **Domain** (values of x): goes from -3 to 5. Hence $-3 \leq x \leq 5$.
- **Range** (values of y): goes from -2 to 4. Hence $-2 \leq y \leq 4$.

> TRY IT
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> What types of graphs are these?

The first is a **cubic**: it has the S-shaped curve, and it crosses the x-axis 3 times (three roots). The second is a **quadratic**: it has the U-shape of a parabola. A quadratic has at most two roots, and a cubic has at most three.

Differentiation

> NOTE
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> **Extended only.** This topic is in the Extended syllabus, so it is asked in Papers 2 and 4. If you sit the Core papers you do not need it.

The gradient of a curve

A straight line has the same gradient everywhere. A curve's gradient **changes**, so we talk about the gradient **at a point**: the gradient of the **tangent** (the straight line that just touches the curve there). You can estimate it by drawing a tangent and finding its gradient with a gradient triangle.

Differentiation finds the gradient exactly. The result is the **gradient function** $\frac{dy}{dx}$, also written $f'(x)$.

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$. Multiply by the power, then reduce the power by 1.

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Function	Derivative
$y = x^3$	$\frac{dy}{dx} = 3x^2$
$y = 5x^2$	$\frac{dy}{dx} = 10x$
$y = 4x$	$\frac{dy}{dx} = 4$
$y = 7$ (a constant)	$\frac{dy}{dx} = 0$
$y = 2x^3 - 6x^2 + 5x - 1$	$\frac{dy}{dx} = 6x^2 - 12x + 5$

Differentiate each term separately. A term in x becomes a constant, and a constant term disappears.

TRY IT

Find the gradient of $y = x^3 - 4x$ when $x = 2$.

$\frac{dy}{dx} = 3x^2 - 4$. When $x = 2$: $3(4) - 4 = 8$.

Tangents

To find the equation of the tangent at a point:

1. Find $\frac{dy}{dx}$ and put in the x-value to get the **gradient** m.
2. Find the y-value of the point from the curve's equation.
3. Use $y - y_1 = m(x - x_1)$ (or $y = mx + c$ and substitute the point).

Stationary (turning) points

At a **stationary point** the tangent is horizontal, so the gradient is zero.

Stationary points: solve $\frac{dy}{dx} = 0$. Then put each x-value into y to get the point.

To decide whether a stationary point is a **maximum** or a **minimum** you can use any method:

- **Second derivative**: if $\frac{d^2y}{dx^2} > 0$ it is a minimum, and if it is negative it is a maximum.

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- **Gradient test**: look at the gradient just before and after the point (negative then positive is a minimum, positive then negative is a maximum).
- **Shape**: a graph of $y = x^2 + \ldots$ (positive x^2) is U-shaped, so its turning point is a minimum.

TRY IT

Find and classify the stationary point of $y = x^2 - 6x + 5$.

$\frac{dy}{dx} = 2x - 6 = 0$, so $x = 3$ and $y = 9 - 18 + 5 = -4$. The point is $(3, -4)$. The second derivative is 2, which is positive, so it is a **minimum**.

Note Academy